THE TRUE ORDER OF THE RIEMANN ZETA–FUNCTION ON THE CRITICAL LINE

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For the Riemann zeta-function on the critical line the terminal estimate have been proved, which had been conjectured by Lindelöf at the beginning of this Centure. The proof is based on the authors relations which connect the bilinear forms of the eigenvalues of the Hecke operators with sums of the Kloosterman sums. By the way, it is proved that for the Hecke series (which are associated with the eigenfunctions of the automorphic Laplacian) the natural analogue of the Lindelöf conjecture is true also. Bibl. 14.

Key words: the Riemann zeta-function, the Kuznetsov trace formulas, the Hecke series

§ 0. PRELIMINARIES

0.1. The main result.

One of the two main problems in the theory of the Riemann zeta-function is the question: what is the order of this function on the critical line?

In this work I prove the following assertion: the Lindelöf conjecture for the Riemann zeta-function is true. Moreover the natural analogue of this conjecture for the Hecke series is true also.

It means that we have the following two theorems.

Theorem 1. *Let $\zeta(s)$ be the Riemann zeta–function which is defined for* $\operatorname{Re} s > 1$ *by two equalities*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{0.1}$$

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where p runs over all primes. Then for any $\varepsilon > 0$

$$|\zeta(1/2+it)| \ll t^{\varepsilon} \tag{0.2}$$

as $t \to +\infty$.

Theorem 2. *Let $\mathcal{H}_j(s)$ be the Hecke series which corresponds to j–th eigenfunction of the automorphic Laplacian for the case of the full modular group. Then for any fixed $j \geqslant 1$ and for any $\varepsilon > 0$ we have*

$$|\mathcal{H}_j(1/2+it)| \ll t^{\varepsilon} \tag{0.3}$$

as $t \to +\infty$.

§ 1. INITIAL IDENTITIES

To prove (0.2) and (0.3) I use the following known facts.

1.1. The fore–traces.

I restrict myself by the case of the full modular group Γ.

Let $\lambda_0 = 0 < \lambda_1 < \ldots \leqslant \lambda_j \leqslant \ldots$ are the eigenvalues of the automorphic Laplacian $\mathcal{L} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. It means for $\lambda = \lambda_j$ there is non–zero solution of the equation

$$\mathcal{L}u = \lambda u \tag{1.1}$$

with the conditions $u(\gamma z) \equiv u\left(\frac{az+b}{cz+d}\right) = u(z)$ for any $\gamma \in \Gamma$ and

$$(u,u) \equiv \int\limits_{\Gamma\backslash\mathbb{H}} |u(z)|^2 d\mu(z) < \infty$$

(here $d\mu(z) = y^{-2}dx\,dy$ is the Γ–invariant measure on the upper half plane $\mathbb{H}$; $\Gamma\backslash\mathbb{H}$ is the fundamental domain of the full modular group Γ).

The continuous spectrum of this boundary problem lies on the half–axis $\lambda \geqslant 1/4$; this spectrum is simple and the corresponding eigenfunction is the analytical continuation of the Eisenstein series $E(z,s)$. This series is defined by the equality

$$E(z,s) = \sum_{\gamma\in\Gamma/\Gamma_\infty} (\operatorname{Im}\gamma z)^s \tag{1.2}$$

for $z \in \mathbb{H}$ and $\operatorname{Re} s > 1$; here Γ_∞ is the syclic subgroup which is generating by the transformation $z \mapsto z+1$.

For all s we have the absolutely convergent series (the Fourier expansion).

Theorem 1.1. *(A. Selberg, S. Chowla [1]). Let $z = x + iy$, $y > 0$; then*

$$E(z,s) = y^s + \frac{\xi(1-s)}{\xi(s)}y^{1-s} + \frac{2}{\xi(s)}\sum_{n\neq 0}\tau_s(n)\mathrm{e}(nx)\sqrt{y}K_{s-1/2}(2\pi|n|y) \tag{1.3}$$

where $\mathrm{e}(x) = \mathrm{e}^{2\pi ix}$,

$$\tau_s(n) = \sum_{d|n,d>0}\left(\frac{|n|}{d^2}\right)^{s-1/2}, \tag{1.4}$$

$$\xi(s) = \pi^{-s}\Gamma(s)\zeta(2s) \tag{1.5}$$

and $K_\nu(\cdot)$ is the modified Bessel function (the Mcdonald function, which is decreasing exponentially at $+\infty$) of the order ν.

Each eigenfunction u_j of the discret spectrum has the similar Fourier expansion, but without zeroth term:

$$u_j(z) = \sqrt{y}\sum_{n\neq 0}\rho_j(n)\mathrm{e}(nx)K_{i\varkappa_j}(2\pi|n|y) \tag{1.6}$$

Here $\rho_j(n)$ are the Fourier coefficients of u_j and for $j \geqslant 1$

$$\varkappa_j = \sqrt{\lambda_j - 1/4}. \tag{1.7}$$

Note that in the case of the full modular group $\lambda_1 \approx 91.14$ (it is the result of the computer calculations; see [14], p. 650-654).

We choose u_j be real and each eigenfunction is even or odd under the reflection operator $\big(T_{-1}f\big)(z) = f(-\overline{z})$; so we have

$$T_{-1}u_j = \varepsilon_j u_j \tag{1.8}$$

with $\varepsilon_j = +1$ or $\varepsilon_j = -1$.

Furthermore, it is possible take these eigenfunctions by such way that they are the eigenfunctions for all the Hecke operators.

Let us define the n–th Hecke operator $T(n)$ by the equality

$$\big(T(n)f\big)(z) = \frac{1}{\sqrt{n}}\sum_{\substack{ad=n\\ d>0}}\ \sum_{b(\mathrm{mod}\, d)} f\left(\frac{az+b}{d}\right); \tag{1.9}$$

then we have for all integers $n, m \geqslant 1$

$$T(n)T(m) = \sum_{d|(n,m)} T\left(\frac{nm}{d^2}\right) = T(m)T(n). \tag{1.10}$$

We take the common system of the eigenfunctions for $\mathcal{L}$ and for all $T(n)$, $n \geqslant 1$. In this case we have for all $n \geqslant 1$ and $j \geqslant 1$

$$\rho_j(n) = \rho_j(1)t_j(n), \quad \rho_j(-n) = \varepsilon_j \rho_j(1)t_j(n), \tag{1.11}$$

where $t_j(n)$ are such (real) numbers that

$$T(n)u_j = t_j(n)u_j. \tag{1.12}$$

We have the following splendid expressions for the bilinear forms of these eigenvalues $t_j(n)$.

Theorem 1.2. *(N. Kuznetsov [2], R. Bruggeman [3]). Let $h(r)$ be an even function in r which is regular in the strip $|\operatorname{Im} r| \leqslant \Delta$ for some $\Delta > 1/2$ and wich is $\mathrm{O}\left(|r|^{-2-\delta}\right)$ for some $\delta > 0$ when $r \to \infty$ inside of this strip. Then for any integers $n, m \geqslant 1$*

$$\sum_{j \geqslant 1} \alpha_j t_j(n) t_j(m) h(\varkappa_j) + \frac{1}{\pi} \int_{-\infty}^{\infty} \tau_{1/2+ir}(n)\tau_{1/2+ir}(m) \frac{h(r)}{|\zeta(1+2ir)|^2} dr =$$

$$= \frac{1}{2\pi}\delta_{n,m} \int_{-\infty}^{\infty} h(u)\, d\chi(u) + \sum_{c \geqslant 1} \frac{1}{c} S(n,m;c)\varphi\left(\frac{4\pi\sqrt{nm}}{c}\right), \tag{1.13}$$

where

$$\alpha_j = (\cosh \pi\varkappa_j)^{-1}|\rho_j(1)|^2, \tag{1.14}$$

$$d\chi(u) = \frac{2}{\pi} u \, \tanh(\pi u)\, du, \tag{1.15}$$

S denotes the Kloosterman sum,

$$S(n,m;c) = \sum_{\substack{a (\operatorname{mod} c) \\ ad \equiv 1 (\operatorname{mod} c)}} \mathrm{e}\left(\frac{na+md}{c}\right), \tag{1.16}$$

and with notation ($J_\nu(\cdot)$– the Bessel function of the order ν)

$$k_0(x,\nu)=\frac{1}{2\cos(\pi\nu)}\left(J_{2\nu-1}(x)-J_{1-2\nu}(x)\right) \tag{1.17}$$

the weight function in the sum of the Kloosterman sums is defined by the integral transform

$$\varphi(x)=\int\limits_{-\infty}^{\infty}k_0(x,1/2+ir)h(r)\,d\chi(r). \tag{1.18}$$

The identity (1.13) is called "the Kuznetsov trace formula"; I think the more preferable say "fore–trace". In the reality the famous Selberg trace formula (for the full modular group and for congruence subgroups) follows from (1.13). So the set of these identities with all $n,m\geqslant 1$ may be considered as the set of primary equalities to construct the trace formulae (and a lot of others identities).

1.2. The regular case.

The first example of the similar identities is the classical Peterson formula.

Theorem 1.3. *(H. Peterson). Let $f_{j,k}$ be the orthogonal Hecke basis in the space $\mathcal{M}_k$ of cusp forms of an even weight k, $\nu_k=\dim\mathcal{M}_k$ and $t_{j,k}(n)$ are the eigenvalues of the Hecke operators $T_k(n):\ \mathcal{M}_k\to\mathcal{M}_k$ under the normalization*

$$\big(T_k(n)f\big)(z)=n^{(k-1)/2}\sum_{\substack{d>0\\ ad=n}}\frac{1}{d^k}\sum_{b(\mathrm{mod}\,d)}f\left(\frac{az+b}{d}\right). \tag{1.19}$$

Then we have

$$\sum_{j=1}^{\nu_k}\|f_{j,k}\|^{-2}t_{j,k}(n)t_{j,k}(m)=\delta_{n,m}+2\pi i^{-k}\sum_{c\geqslant 1}\frac{1}{c}S(n,m;c)J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right). \tag{1.20}$$

Note that for $k=2,4,6,8,10$ and 14 the sum on the left side (1.20) is zero, since $\nu_k=[k/12]$ if $k\not\equiv 2(\mathrm{mod}12)$ and $\nu_k=\left[\frac{k}{12}\right]-1$ for $k\equiv 2(\mathrm{mod}12)$ (here $[x]$ denotes the integral part of x).

1.3. The sum of the Kloosterman sums.

We can invert (1.13) $\big($and the similar identity with $\varepsilon_j\alpha_j$ instead of α_j which is expessed in terms of $S(n,-m;c)\big)$ and we shall assume that the sum of Kloosterman sums is given rather than the bilinear form in the Fourier coefficients.

Theorem 1.4. *(N. Kuznetsov [2],[4]). Let $\varphi \in \mathrm{C}^3(0,\infty)$, $\varphi(0) = \varphi'(0) = 0$ and assume that $\varphi(x)$ together with its derivatives up to third order is $\mathrm{O}(x^{-\beta})$ for some $\beta > 2$ as $x \to +\infty$. Then, for any integers $n, m \geqslant 1$, we have*

$$\sum_{c\geqslant 1} \frac{1}{c} S(n,m;c)\varphi\left(\frac{4\pi\sqrt{nm}}{c}\right) = \sum_{j\geqslant 1} \alpha_j t_j(n) t_j(m) h(\varkappa_j) + \\ + \frac{1}{\pi}\int_{-\infty}^{\infty} \tau_{1/2+ir}(n)\tau_{1/2+ir}(m)\frac{h(r)}{|\zeta(1+2ir)|^2}dr + \\ + \sum_{k\geqslant 6} g(k) \sum_{1\leqslant j\leqslant \nu_{2k}} \alpha_{j,2k} t_{j,2k}(n) t_{j,2k}(m), \quad (1.21)$$

where h and g are defined in terms of φ by the integral transforms

$$h(r) = \pi \int_0^{\infty} k_0(x, 1/2+ir)\varphi(x)\frac{dx}{x}, \quad (1.22)$$

$$g(k) = (2k-1)\int_0^{\infty} J_{2k-1}(x)\varphi(x)\frac{dx}{x} \quad (1.23)$$

and $\alpha_{j,2k}$ is written instead of $\|f_{j,2k}\|^{-2}$.

The proof may be found as well as in [4] and (in more general situation) in [5],[11].

Theorem 1.5. *(N. Kuznetsov[4], M. Huxley [5]). Assume that to a function $\psi:\ [0,\infty) \to \mathbb{C}$ the integral transform*

$$h(r) = 2\cosh(\pi r)\int_0^{\infty} K_{2ir}(x)\psi(x)\frac{dx}{x} \quad (1.24)$$

associates the function h satisfying to the conditions of Theorem 1.2. Then, for this ψ and for integers $n, m \geqslant 1$, we have

$$\sum_{c\geqslant 1} \frac{1}{c} S(n,-m;c)\psi\left(\frac{4\pi\sqrt{nm}}{c}\right) = \sum_{j\geqslant 1} \varepsilon_j \alpha_j t_j(n) t_j(m) h(\varkappa_j) + \\ + \frac{1}{\pi}\int_{-\infty}^{\infty} \tau_{1/2+ir}(n)\tau_{1/2+ir}(m)\frac{h(r)}{|\zeta(1+2ir)|^2}dr. \quad (1.25)$$

Note that (1.24) is true if h is satisfying to conditions of Theorem 1.2 and, for a given h, the taste function on the left side is defined by the integral transform

$$\psi(x) = \frac{4}{\pi^2} \int\limits_{-\infty}^{\infty} K_{2ir}(x) h(r)\, r\, \sinh(\pi r) dr. \tag{1.26}$$

The pair of transforms (1.24) and (1.26), which are inverse to each other, is the Kontorovich–Lebedev transform.

1.4. The Hecke series.

The Hecke series $\mathcal{H}_j(s)$ and $\mathcal{H}_{j,k}(s)$ are defined for $\operatorname{Re} s > 1$ as

$$\mathcal{H}_j(s) = \sum_{n=1}^{\infty} n^{-s} t_j(n), \quad \mathcal{H}_{j,k}(s) = \sum_{n=1}^{\infty} n^{-s} t_{j,k}(n). \tag{1.27}$$

Here $t_j(n)$ and $t_{j,k}(n)$ are the eigenvalues of the Hecke operators in the space of the cusp forms of the weight zero and the even positive k correspondingly.

The Hecke series which corresponds to continuous spectrum of the automorphic Laplacian is

$$\mathcal{L}_\nu(s,x) = \sum_{n=1}^{\infty} n^{-s} \tau_\nu(n) \mathrm{e}(nx), \quad \mathrm{e}(x) = \mathrm{e}^{2\pi i x}. \tag{1.28}$$

The following assertions are well known.

Theorem 1.6. *The Hecke series $\mathcal{H}_j(s)$ and $\mathcal{H}_{j,k}(s)$ are the entire functions and these series have the functional equations of the Riemann type:*

$$\mathcal{H}_j(s) = (4\pi)^{2s-1} \gamma(1-s, 1/2 + i\varkappa_j)\big(-\cos(\pi s) + \varepsilon_j \cosh(\pi\varkappa_j)\big)\mathcal{H}_j(1-s), \tag{1.29}$$

$$\mathcal{H}_{j,k}(s) = -(4\pi)^{2s-1} \gamma(1-s, k/2) \cos(\pi s) \mathcal{H}_{j,k}(1-s) \tag{1.30}$$

where

$$\gamma(u,v) = \frac{1}{\pi} 2^{2u-1} \Gamma(u+v-1/2)\Gamma(u-v+1/2). \tag{1.31}$$

Theorem 1.7. *Let x be rational, $x = \frac{d}{c}$ with $(d, c) = 1,\ \ c \geqslant 1$. Then for $\nu \neq 1/2$ the only singularities of $\mathcal{L}_\nu(s, d/c)$ are simple poles at the point $s_1 = \nu + 1/2$ and $s_2 = 3/2 - \nu$ with residues $c^{-2\nu}\zeta(2\nu)$ and $c^{2\nu-2}\zeta(2-2\nu)$; the function $\mathcal{L}(s, d/c)$ has the functional equation*

$$\mathcal{L}_\nu\left(s, \frac{d}{c}\right) = \left(\frac{4\pi}{c}\right)^{2s-1} \gamma(1-s, \nu)\times$$
$$\times \left\{ -\cos(\pi s)\mathcal{L}_\nu\left(1-s, -\frac{a}{c}\right) + \sin(\pi\nu)\mathcal{L}_\nu\left(1-s, -\frac{a}{c}\right)\right\}, \quad (1.32)$$

where a is defined by the congruence

$$ad \equiv 1 (\mathrm{mod} c). \quad (1.33)$$

1.5. The Ramanujan identities.

Let

$$\mathcal{Z}(s; \nu, \mu) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\tau_\nu(n)\tau_\mu(n)}{n^s}; \quad (1.34)$$

this series converges absolutely if $\mathrm{Re}\, s > 1 + |\mathrm{Re}(\nu - 1/2)| + |\mathrm{Re}(\mu - 1/2)|$. The Ramanujan identity ([12], 1.3) gives the explicit expression for this function.

Theorem 1.8.

$$\mathcal{Z}(s; \nu, \mu) = \zeta(s + \nu - \mu)\zeta(s - \nu + \mu)\zeta(s + \nu + \mu - 1)\zeta(s - \nu - \mu + 1). \quad (1.35)$$

The analogues of this identity for the Hecke series are the consequence of the multiplicative relations (1.10) for Hecke operators.

Theorem 1.9. *Let* $\mathrm{Re}\, s > 1 + |\mathrm{Re}(\nu - 1/2)|$*; then*

$$\zeta(2s) \sum_{n=1}^{\infty} n^{-s}\tau_\nu(n)t_j(n) = \mathcal{H}_j(s + \nu - 1/2)\mathcal{H}_j(s - \nu + 1/2), \quad (1.36)$$

$$\zeta(2s) \sum_{n=1}^{\infty} n^{-s}\tau_\nu(n)t_{j,k}(n) = \mathcal{H}_{j,k}(s + \nu - 1/2)\mathcal{H}_{j,k}(s - \nu + 1/2). \quad (1.37)$$

§ 2. FUNCTIONAL EQUATION FOR THE FOURTH MOMENTS

We define the fourth moments of the Hecke series by the equalities

$$Z^{(d)}(s,\nu;\rho,\mu|h_0,h_1)=\sum_{j\geqslant 1}\alpha_j\big(h_0(\varkappa_j)+\varepsilon_j h_1(\varkappa_j)\big)\times \\ \times\mathcal{H}_j(s+\nu-1/2)\mathcal{H}_j(s-\nu+1/2)\mathcal{H}_j(\rho+\mu-1/2)\mathcal{H}_j(\rho-\mu+1/2), \quad (2.1)$$

$$Z^{(r)}(s,\nu;\rho,\mu|g)= \\ =\sum_{k\geqslant 6}g(k)\sum_{1\leqslant j\leqslant\nu_{2k}}\alpha_{j,2k}\mathcal{H}_{j,2k}(s+\nu-1/2)\mathcal{H}_{j,2k}(s-\nu+1/2)\times \\ \times\mathcal{H}_{j,2k}(\rho+\mu-1/2)\mathcal{H}_{j,2k}(\rho-\mu+1/2). \quad (2.2)$$

It would be assumed here and further that $\operatorname{Re}\nu=\operatorname{Re}\mu=1/2$; then the first series is the result of double summation

$$Z^{(d)}(s,\nu;\rho,\mu|h_0,h_1)=\zeta(2s)\zeta(2\rho)\sum_{n,m\geqslant 1}\frac{\tau_\nu(n)\tau_\mu(m)}{n^s m^\rho}\times \\ \times\left(\sum_{j\geqslant 1}\alpha_j\left(h_0(\varkappa_j)+\varepsilon_j h_1(\varkappa_j)\right)t_j(n)t_j(m)\right), \quad (2.3)$$

if $\operatorname{Re}s,\ \ \operatorname{Re}\rho>1$; the function $Z^{(r)}(s,\nu;\rho,\mu|g)$ is defined by the similar way for these values of s,ρ.

The corresponding moment for the continuous spectrum is

$$Z^{(c)}(s,\nu;\rho,\mu|h)=\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\mathcal{Z}(s;\nu,1/2+ir)\mathcal{Z}(\rho;\mu,1/2+ir)\frac{h(r)}{|\zeta(1/2+ir)|^2}dr \quad (2.4)$$

with the condition $\operatorname{Re}s,\operatorname{Re}\rho<1$ (for $\operatorname{Re}\nu=\operatorname{Re}\mu=1/2$); here $\mathcal{Z}$ is defined by (1.35).

The same integral (2.4) with $\operatorname{Re}s,\operatorname{Re}\rho>1$ we denote by $Z^{(c,+)}(s,\nu;\rho,\mu|h)$.

The main purpose of this section is the correction of the proof of Theorem 15 from [6].

2.1. The Mellin transform for the integral (1.18).

Let for a given h the function φ is defined by the integral transform (1.18).

First of all I give the explicit form of the Mellin transform for this φ.

Proposition 2.1. *Let $h(r)$ be the even function which is regular in the strip $|\mathrm{Im}\ r| \leqslant \Delta, \quad \Delta > 1/2$, and $|h(r)|$ decreases faster than any fixed degree of r when $r \to \infty$ in this strip. If φ is defined by the integral (1.18), then the Mellin transform of φ is given by the equality*

$$\widehat{\varphi}(2w) \equiv \int_0^\infty \varphi(x)x^{2w-1}dx = \frac{1}{\pi}2^{2w-1}\cos(\pi w)\int_{-\infty}^{\infty}\Gamma(w+iu)\Gamma(w-iu)h(u)\,d\chi(u). \tag{2.5}$$

This function is regular in half–plane $\mathrm{Re}\, w > -\Delta$ *excepting simple poles at* $w = -1/2, -3/2, -5/2, \dots$ *and for any fixed* $\mathrm{Re}\, w$ *we have*

$$\widehat{\varphi}(2w) = 2^{2w}\cos(\pi w)w^{2w-1}\mathrm{e}^{-2w}\int_{-\infty}^{\infty}\left(1+\frac{p_1(iu)}{w}+\frac{p_2(iu)}{w^2}+\dots\right)h(u)\,d\chi(u) \tag{2.6}$$

as $w \to \infty$; here $p_1, p_2, \dots$ are polynomials of degree $2, 4, \dots$,

$$p_1(z) = z^2+\frac{1}{6},\ p_2(z) = \frac{1}{2}z^4+\frac{2}{3}z^2+\frac{1}{72},\ p_3(z) = \frac{1}{6}z^6+\frac{5}{4}z^4+\frac{5}{24}z^2+\frac{1}{1620},\dots \tag{2.7}$$

To prove these assertions we assume firstly that $\mathrm{Re}\, w \in (0, 1/4)$. For this case we can integrate term by term and the known table integrals give the equality (2.5) for $0 < \mathrm{Re}\, w < 1/4$.

The regularity $\widehat{\varphi}(2w)$ for $\mathrm{Re}\, w > 0$ is obvious. In the strip $-1 < \mathrm{Re}\, w < 0$ we have

$$\widehat{\varphi}(2w) = -\frac{1}{\pi}2^{2w+2}w\sin(\pi w)\Gamma(2w)h(iw)+ \\ +\frac{2^{2w}}{2\pi}\cos(\pi w)\int_{-\infty}^{\infty}\Gamma(w+iu)\Gamma(w-iu)h(u)\,d\chi(u). \tag{2.7}$$

Let $|\mathrm{Re}\, w|$ be small and $\mathrm{Re}\, w > 0$. Move the path of integration in (2.5) to the new path $\mathcal{C}$ made up of

$\mathcal{C}_1$: line segment $-\infty$ to $-|\operatorname{Im} w| - \delta, \delta > \operatorname{Re} w$,
$\mathcal{C}_2$: anticlockwise semicircle centre $-|\operatorname{Im} w|$ radius δ, such that the point $-iw$ is lying above the path,
$\mathcal{C}_3$: line segment $-|\operatorname{Im} w| + \delta$ to $|\operatorname{Im} w| - \delta$,
$\mathcal{C}_4$: clockwise semicircle centre $|\operatorname{Im} w|$ radius δ (the point iw is lying below the path),
$\mathcal{C}_5$: line segment $|\operatorname{Im} w| + \delta$ to $+\infty$.

For $\operatorname{Re} w > 0$ we have

$$\hat{\phi}(2w) = -2\pi i \mathrm{Res}_{u=-iw} + 2\pi i \mathrm{Res}_{u=iw} + \int\limits_{\mathcal{C}} (\ldots) du.$$

Note that the sum of residues at the points $u = \pm iw$ is the first term in (2.7).

But the integral over $\mathcal{C}$ is regular for $-1 < \operatorname{Re} w < 0$ and for the case $\operatorname{Re} w < 0$ we can integrate over the real axis again; it gives us the equality (2.7).

By the similar way we can receive the analytical continuation in the strip $-2 < \operatorname{Re} w < -1$ (if $\Delta > 1$) and so on; so $\widehat{\varphi}(2w)$ is regular in the half–plane $\operatorname{Re} w > -\Delta$ excepting those poles of $\Gamma(2w)$ which are not compensated by zeroes of $\sin \pi w$.

After that we use the Barnes asymptotic expansion([7], 1.18). Namely, let $B_n(z)$ are Bernoulli polynomials ($B_0 = 1,\ B_1 = z - 1/2,\ B_2 = z^2 - z + 1/6,\ B_3 = z^3 - \frac{3}{2}z^2 + \frac{1}{2}z, \ldots$). The Barnes expansion is the asymptotic series

$$\log \Gamma(w+z) = (w+z-1/2)\log w - w + \frac{1}{2}\log(2\pi) + Q(w,z) \tag{2.8}$$

with

$$Q(w,z) = \frac{B_2(z)}{1\cdot 2\, w} - \frac{B_3(z)}{2\cdot 3\, w^2} + \ldots + \frac{(-1)^{n+1}B_{n+1}(z)}{n(n+1)w^n} + \ldots\ ; \tag{2.9}$$

this expansion holds if $w \to \infty$, $|\arg w| < \pi$ and $z = o\left(|w|^{1/2}\right)$.

For any fixed M and for $u_0 = |w|^{\varepsilon}$ with the arbitrary small (but fixed) $\varepsilon > 0$ our integral (2.5) equals to

$$\int\limits_{|u| \leqslant u_0} (\ldots)\, d\chi(u) + O(|w|^{-M}),$$

since we assume the very fast decreasing of h.

For $|u| \leqslant u_0$ we can use the Barnes expansion; it gives the asymptotic equality

$$\Gamma(w+z)\Gamma(w-z)=2\pi w^{2w-1}\mathrm{e}^{-2w}\exp\Big(Q(w,z)+Q(w,-z)\Big)=$$
$$=2\pi w^{2w-1}\mathrm{e}^{-2w}\cdot\left(1+Q(w,z)+Q(w,-z)+\frac{1}{2!}\Big(Q(w,z)+Q(w,-z)\Big)^2+\ldots\right).$$

Taking the finite number of terms from this expansion we write integral in the limits $(-\infty,+\infty)$ again and come to (2.6).

2.2 The regularization of the initial identity.

It follows from (2.6) that in the general case the Mellin transform (2.5) of the integral (1.18) is $O(|w|^{2\mathrm{Re}\,w-1})$ as $w\to\infty$ with the fixed value of $\mathrm{Re}\,w$, excepting some cases when $h(u)$ is orthogonal to the degrees $u^m, 0\leqslant m\leqslant m_0$ on the mesure $d\chi(u)$.

Nevertherless, there is the method of "regularization" of this φ. We substract from the integral (1.18) the combination of the Bessel functions; the coefficients of this combination may be determined so that the difference has the Mellin transform which decreases more rapidly.

This regularization is very essential for our proof and it will be used later many times.

Proposition 2.2. *Let $L=\{l_1,\ldots,l_N\}$ be finite set of integers, $1\leqslant l_1<l_2<\ldots<l_N$, $N\geqslant 1$. Let the function h be taken under assumptions of Proposition 2.1 with the additional conditions $h(i(l-1/2))=0$ for $1\leqslant l\leqslant \Delta-1/2$.*

We define N coefficients $c(l)$ from the linear system

$$\sum_{l\in L}(l-1/2)^{2m}(-1)^l c(l)=(-1)^m\int_{-\infty}^{\infty}u^{2m}h(u)\,d\chi(u),\qquad 0\leqslant m\leqslant N-1. \quad (2.10)$$

Let

$$\Phi_N(x)=\varphi(x)-\sum_{l\in L}c(l)J_{2l-1}(x), \quad (2.11)$$

where φ is defined by the integral transform (1.18) of this h; then the Mellin transform of this difference,

$$\widehat{\Phi}_N(2w)=\int_0^{\infty}\Phi_N(x)x^{2w-1}\,dx, \quad (2.12)$$

is the regular function in the half plane $\operatorname{Re} w > \max(-\Delta, -l_1 + 1/2)$ *and for any fixed* $\operatorname{Re} w$ *we have*

$$\left|\widehat{\Phi}_N(2w)\right| \ll |w|^{2\operatorname{Re} w - N - 1} \tag{2.13}$$

as $w \to \infty$.

Really,

$$\int_0^\infty J_{2l-1}(x)x^{2w-1}dx = 2^{2w-1}\frac{\Gamma(l-1/2+w)}{\Gamma(l+1/2-w)} =$$
$$= \frac{1}{\pi}2^{2w-1}(-1)^l\cos(\pi w)\Gamma(w+l-1/2)\Gamma(w-l+1/2).$$

Now we have for large $|w|$ (and fixed $\operatorname{Re} w$)

$$\begin{aligned}\widehat{\Phi}_N(2w) = 2^{2w}\cos(\pi w)w^{2w-1}\mathrm{e}^{-2w}\times\\ \times\left\{\int_{-\infty}^{\infty}\left(1+\frac{p_1(iu)}{w}+\frac{p_2(iu)}{w^2}+\dots\right)h(u)\,d\chi(u)-\right.\\ \left.-\sum_{l\in L}(-1)^l c(l)\left(1+\frac{p_1(l-1/2)}{w}+\frac{p_2(l-1/2)}{w^2}+\dots\right)\right\}\end{aligned} \tag{2.14}$$

(note that two line segments $|u| \geqslant |w|^\varepsilon$ for any fixed $\varepsilon > 0$ give $O(|w|^{-M})$ for any $M \geqslant 6$; so the Barnes expansion is used for $|u| \leqslant |w|^{varepsilon}$ only).

If $c(l)$ are defined by (2.10) then the terms w^{-k} with $0 \leqslant k \leqslant N-1$ are cancelled. It gives the estimate (2.13).

Now we rewrite the initial identity (1.13).

Proposition 2.3. *Let* $h(r)$ *be an even function in* r, *regular in the strip* $|\operatorname{Im} r| \leqslant \Delta$ *with some* $\Delta > 3/2$ *and let* $|h(r)|$ *decreases faster than any fixed degree* $|r|$ *as* $r \to \infty$ *in this strip; in addition we assume* $h\big((l-1/2)i\big) = 0$ *for* $1 \leqslant l \leqslant \Delta + 1/2$. *Let* Φ_N *be defined by (2.11) with coefficients from (2.10); then for any integers*

$n, m \geqslant 1$ we have

$$\sum_{j\geqslant 1}\alpha_j t_j(n)t_j(m)h(\varkappa_j)+\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\tau_{1/2+ir}(n)\tau_{1/2+ir}(m)\frac{h(r)}{|\zeta(1+2ir)|^2}dr=$$
$$=\sum_{c\geqslant 1}\frac{1}{c}S(n,m;c)\Phi_N\left(\frac{4\pi\sqrt{mn}}{c}\right)+$$
$$+\frac{1}{2\pi}\sum_{l\in L}(-1)^l c(l)\sum_{1\leqslant j\leqslant \nu_{2l}}\alpha_{j,2l}t_{j,2l}(n)t_{j,2l}(m). \quad (2.15)$$

This equality is the immediate consequence of the Peterson identity (1.20). Writing

$$\varphi=\Phi_N+\sum_{l\in L}c(l)J_{2l-1},$$

we get the sum of the Kloosterman sums with the test function Φ_N and the finite number sums where the test function is the Bessel function of odd order. The singular term with $\delta_{n,m}$ contributes

$$-\frac{1}{2\pi}\delta_{n,m}\sum_{l\in L}(-1)^l c(l)=-\frac{1}{2\pi}\delta_{n,m}\int\limits_{-\infty}^{\infty}h(u)\,d\chi(u). \quad (2.16)$$

(the equation (2.10) with $m=0$). So the term with $\delta_{n,m}$ is disappearing and we come to (2.15).

2.2 The functional equation.

Theorem 2.2. *Let two parameters ν,μ are taken with conditions $\operatorname{Re}\nu=\operatorname{Re}\mu=1/2$, $\nu\neq 1/2$, $\mu\neq 1/2$ and two variables s,ρ are taken inside of the narrow strip $5/4<\operatorname{Re}s,\operatorname{Re}\rho<5/4+\varepsilon$ with small (but fixed) $\varepsilon>0$.*

Let Φ_N for a given h, which is satisfying to all conditions of Proposition 2.3, be defined by (2.11) with $N\geqslant 1$ and $l_1\geqslant 2$ Then we have

$$Z^{(d)}(s,\nu;\rho,\mu|h,0)+Z^{(c,+)}(s,\nu;\rho,\mu|h,0)=$$
$$=Z^{(d)}(\rho,\nu;s,\mu|h_0,h_1)+Z^{(c,+)}(\rho,\nu;s,\mu|h_0,h_1)+Z^{(r)}(\rho,\nu;s,\mu|g)+$$
$$+\frac{1}{2\pi}\sum_{l\in L}(-1)^l c(l)z_{2l}(s,\nu;\rho,\mu)+\mathcal{R}_h(s,\nu;\rho,\mu)+\mathcal{R}_h(s,1-\nu;\rho,\mu)+$$
$$+\mathcal{R}_h(\rho,\mu;s,\nu)+\mathcal{R}_h(\rho,1-\mu;s,\nu) \quad (2.17)$$

where z_{2l} are defined by (2.27), $c(l)$ are taken from (2.10),

$$\mathcal{R}_h(s,\nu;\rho,\mu)=2(4\pi)^{2s-2\nu-1}\widehat{\Phi}_N(2\nu+1-2s)\cdot\frac{\zeta(2\rho)\zeta(2\nu)}{\zeta(2\rho+2\nu)}\mathcal{Z}(\rho+\nu;s,\mu) \tag{2.18}$$

and the coefficients h_0, h_1, g are defined by equalities

$$\begin{aligned} h_0(r)\equiv h_0(r;s,\nu,\rho,\mu)=&\\ &=-i\int\limits_{(\Delta)}\gamma(w,1/2+ir)\gamma(\rho-w,\nu)\gamma(s-w,\mu)\cos\pi w\times\\ &\times\Big(\cos\pi(\rho-w)\cos\pi(s-w)+\sin\pi\nu\sin\pi\mu\Big)\widehat{\Phi}_N(2w-2s-2\rho+2)dw, \end{aligned} \tag{2.19}$$

$$\begin{aligned} h_1(r)\equiv h_1(r;s,\nu,\rho,\mu)=&\\ &=-i\int\limits_{(\Delta)}\gamma(w,1/2+ir)\gamma(\rho-w,\nu)\gamma(s-w,\mu)\mathrm{ch}(\pi r)\times\\ &\times\widehat{\Phi}_N(2w-2s-2\rho+2)\Big(\cos\pi(s-w)\sin\pi\nu+\cos\pi(\rho-w)\sin\pi\mu\Big)dw, \end{aligned} \tag{2.20}$$

$$\begin{aligned} g(k)\equiv g(k;s,\nu,\rho,\mu)=&\\ &=\frac{2(2k-1)}{\pi i}\int\limits_{(\Delta)}\frac{\Gamma(k-1/2+w)}{\Gamma(k+1/2-w)}2^{2w-1}\gamma(\rho-w,\nu)\gamma(s-w,\mu)\times\\ &\times\widehat{\Phi}_N(2w-2s-2\rho+2)\Big(\cos\pi(\rho-w)\cos\pi(s-w)+\sin\pi\nu\sin\pi\mu\Big)dw, \end{aligned} \tag{2.21}$$

where Δ is taken with condition $0<\Delta<\min(\mathrm{Re}\, s,\mathrm{Re}\,\rho)$.

Before proving of this identity I give the following simple estimate.

Proposition 2.4. *Let s and ν are fixed and $\sigma=\mathrm{Re}\, s\in(1/2,1)$, $\mathrm{Re}\nu=1/2$; then for any d with condition $(d,c)=1$ we have for any $\varepsilon>0$*

$$\left|\mathcal{L}_\nu\left(s,\frac{d}{c}\right)\right|\ll c^{1-\sigma+\varepsilon}, \tag{2.22}$$

as an integer $c \to +\infty$.

If $\operatorname{Re} s \geqslant 1+\varepsilon$ then $\mathcal{L}_\nu(s, \frac{d}{c})$ is bounded uniformly in c.

For the case $\operatorname{Re} s = -\varepsilon$ we have from the functional equation (1.32)

$$|\mathcal{L}_\nu(s, \frac{d}{c})| \ll_{s,\nu} c^{1+2\varepsilon}. \tag{2.23}$$

Now (2.22) follows from the Phragmén-Lindelöf principle.

After this we return to (2.17) and consider the triple sum

$$\zeta(2s)\zeta(2\rho) \sum_{n,m \geqslant 1} \frac{\tau_{\nu(n)}}{n^s} \frac{\tau_{\mu(m)}}{m^\rho} \sum_{c \geqslant 1} \frac{1}{c} S(n,m;c) \Phi_N\left(\frac{4\pi\sqrt{nm}}{c}\right) \tag{2.24}$$

where instead of Φ_N we can write the Mellin integral

$$\Phi_N(x) = \frac{1}{\pi i} \int\limits_{(\delta)} \widehat{\Phi}_N(2w) x^{-2w} dw. \tag{2.25}$$

Under our assumptions $\widehat{\Phi}_N(2w)$ is regular for $\operatorname{Re} w > -\min(\Delta, 3/2)$ and we have the estimate (2.13) with $N \geqslant 1$. As the consequence we have $|\Phi_N(x)| \ll \min\left(x^3, x^{-1+\varepsilon}\right)$ for any $\varepsilon > 0$ (since $\Delta > 3/2$ we can take $\delta = -3/2$ in (2.25) if $x \to 0$; the integral is absolutely convergent for $\operatorname{Re} w < 3/2$ and when $x \to \infty$ we can take $\delta = 1/2 - \varepsilon, \ \ \varepsilon > 0$).

The triple sum (2.24) equals to (we use (2.15))

$$Z^{(d)}(s,\nu;\rho,\mu|h) + Z^{(c,+)}(s,\nu;\rho,\mu|h) - \frac{1}{2\pi} \sum_{l \in L} (-1)^l c(l) z_{2l}(s,\nu;\rho,\mu) \tag{2.26}$$

where

$$z_{2l}(s,\nu;\rho,\mu) = \\ = \sum_{1 \leqslant j \leqslant \nu_{2l}} \alpha_{j,2l} \mathcal{H}_{j,2l}(s+\nu-1/2) \mathcal{H}_{j,2l}(s-\nu+1/2) \mathcal{H}_{j,2l}(\rho+\mu-1/2) \mathcal{H}_{j,2l}(\rho-\mu+1/2) \tag{2.27}$$

On the other side, as the consequence of the Weil estimate, we have for any $\varepsilon > 0$ the majorant

$$\sum_{n,m \geqslant 1} \sum_{c \geqslant 1} n^{-\sigma_1} m^{-\sigma_2} d(m) d(n) d(c) \frac{\sqrt{(n,m,c)}}{\sqrt{c}} \min\left((\frac{\sqrt{nm}}{c})^3, (\frac{c}{\sqrt{nm}})^{1-2\epsilon} \right) \tag{2.28}$$

for the series in (2.24) $\big((n,m,c)$ is the greatest common divisor of n,m,c; and $\sigma_1 = \operatorname{Re} s,\quad \sigma_2 = \operatorname{Re}\rho\,\big)$. This series converges for $\sigma_1, \sigma_2 > 5/4$.

It means the series in (2.24) converges absolutely and we can sum in any order. Doing the summation over n,m in the first line, we get for our triple sum the expression

$$\frac{\zeta(2s)\zeta(2\rho)}{\pi i}\sum_{c\geqslant 1}\frac{1}{c}\sum_{n,m\geqslant 1}\frac{\tau_{\nu(n)}\tau_{\mu(m)}}{n^s m^\rho}S(n,m;c)\int_{(\delta)}\widehat{\Phi}_N(2w)\left(\frac{4\pi\sqrt{nm}}{c}\right)^{-2w}dw. \tag{2.29}$$

Here for any fixed $c\geqslant 1$ we can take (and take) $\delta\in(0,1/2)$; under this condition our integral converges absolutely and the inner double sum is

$$\sum_{\substack{a(\operatorname{mod} c)\\ ad\equiv 1(\operatorname{mod} c)}}\int_{(\delta)}\left(\frac{c}{4\pi}\right)^{2w}\widehat{\Phi}_N(2w)\mathcal{L}_\nu\left(s+w,\frac{a}{c}\right)\mathcal{L}_\mu\left(\rho+w,\frac{d}{c}\right)dw. \tag{2.30}$$

To receive the new expression we move the path of the integration to the left on the line $\operatorname{Re} w=\delta_1$ so that $\sigma_1+\delta_1,\;\; \sigma_2+\delta_1<0$. Four poles at $w=1-s\pm(\nu-1/2)$ and $w=1-\rho\pm(\mu-1/2)$ (where $\mathcal{L}_\nu\left(s+w,\frac{a}{c}\right)$ and $\mathcal{L}_\mu\left(\rho+w,\frac{d}{c}\right)$ have the residues $c^{-2\pm(2\nu-1)}\zeta\big(1\pm(2\nu-1)\big)$ and $c^{-2\pm(2\mu-1)}\zeta\big(1\pm(2\mu-1)\big)$ correspondingly) give us the terms

$$\begin{gathered}
2\zeta(2s)\zeta(2\rho)\sum_{c\geqslant 1}\frac{\zeta(2\nu)(4\pi)^{2s-2\nu-1}}{c^{2s}}\widehat{\Phi}_N(2\nu+1-2s)\sum_{ad\equiv 1(\operatorname{mod} c)}\mathcal{L}_\mu\left(\rho+\mu+1/2-s,\frac{d}{c}\right)+\\
+\{\text{the same with } \nu \text{ replaced by } 1-\nu\}+\\
+2\zeta(2s)\zeta(2\rho)\sum_{c\geqslant 1}\frac{\zeta(2\mu)(4\pi)^{2\rho-2\mu-1}}{c^{2\rho}}\widehat{\Phi}_N(2\mu+1-2\rho)\sum_{ad\equiv 1(\operatorname{mod} c)}\mathcal{L}_\nu\left(s+\mu+1/2-\rho,\frac{a}{c}\right)+\\
+\{\text{the same with } \mu \text{ replaced by } 1-\mu\}.
\end{gathered} \tag{2.31}$$

Under our assumption we have $|\operatorname{Re}(\rho-s)|\leqslant\varepsilon$; so

$$\left|\mathcal{L}_\mu\left(\rho+\nu+1/2-s,\frac{d}{c}\right)\right|\ll c^\varepsilon \tag{2.32}$$

and all series in (2.31) are convergent absolutely and they define the regular functions in ρ and s. If $\mathrm{Re}\,(\rho - s) > 0$ then we have

$$\sum_{(d,c)=1} \mathcal{L}_\mu \left(\rho + \nu + 1/2 - s, \frac{d}{c}\right) = \sum_{n \geqslant 1} \frac{\tau_\mu(n) S(0,n;c)}{n^{\rho+\nu+1/2-s}} \tag{2.33}$$

and

$$\sum_{c \geqslant 1} \frac{1}{c^{2s}} \sum_{(d,c)=1} \mathcal{L}_\mu \left(\rho + \nu + 1/2 - s, \frac{d}{c}\right) = \frac{1}{\zeta(2s)} \mathcal{Z}(\rho + \nu; s, \mu). \tag{2.34}$$

The analitical continuation of these equalities gives four terms with $\widehat{\Phi}_N$ on the right side (2.17). In the integral on the line $\mathrm{Re}\, w = -\delta_1$ we use the functional equation (1.32) and we come to the expression

$$\zeta(2s)\zeta(2\rho) \sum_{c \geqslant 1} \frac{1}{c} \sum_{n,m \geqslant 1} \frac{\tau_{\nu(n)}}{n^\rho} \frac{\tau_{\mu(m)}}{m^s} \times \\ \times \left\{ S(n,m;c) \varphi_0 \left(\frac{4\pi\sqrt{nm}}{c} \right) + S(n,-m;c) \varphi_1 \left(n,-m;c\right) \right\} \tag{2.35}$$

where for $x > 0$ φ_0 and φ_1 are defined by the integrals

$$\varphi_0(x) \equiv \varphi_0(x; s, \nu, \rho, \mu) = \frac{1}{i\pi} x^{2s+2\rho-2} \int_{(\delta_1)} \gamma(1 - s - w, \nu) \gamma(1 - \rho - w, \mu) \times \\ \times \Big(\cos \pi(s + w) \cos \pi(\rho + w) + \sin \pi\nu \sin \pi\mu \Big) \widehat{\Phi}_N(2w) x^{2w} dw, \tag{2.36}$$

$$\varphi_1(x) \equiv \varphi_1(x; s, \nu, \rho, \mu) = -\frac{1}{i\pi} x^{2s+2\rho-2} \int_{(\delta_1)} \gamma(1 - s - w, \nu) \gamma(1 - \rho - w, \mu) \times \\ \times \Big(\cos \pi(\rho + w) \sin \pi\nu + \cos \pi(s + w) \sin \pi\mu \Big) \widehat{\Phi}_N(2w) x^{2w} dw; \tag{2.37}$$

it is possible take here any δ_1 with $-3/2 \leqslant \delta_1 < \min(1 - \mathrm{Re}\, s, 1 - \mathrm{Re}\, \rho)$.

Both integrals (2.36) and (2.37) and the triple series (2.35) converge absolutely if $5/4 < \operatorname{Re} s,\ \operatorname{Re}\rho < 5/4+\varepsilon$ for some small ε (it is assumed $\operatorname{Re}\nu = \operatorname{Re}\mu = 1/2,\ \ \nu \neq 1/2, \mu \neq 1/2$). Really, we have

$$\varphi_j(x) = O\Big(\max\big(x^{2\operatorname{Re} s}, x^{2\operatorname{Re}\rho}\big)\Big) \quad \text{as} \quad x \to 0 \tag{2.38}$$

and for any $\varepsilon_1 > 0$

$$\varphi_j(x) = O\Big(x^{2\operatorname{Re} s + 2\operatorname{Re}\rho - 7 + \varepsilon_1}\Big), \qquad x \to +\infty. \tag{2.39}$$

Since (the Weil estimate again) $|S(n,m;c)| \leqslant c^{1/2} d(c)(m,n,c)^{1/2}$ we have the majorant

$$\frac{d(n)d(m)d(c)}{m^{\sigma_1} n^{\sigma_2}} \cdot \min\left(\left(\frac{\sqrt{mn}}{c}\right)^{\sigma_2}, \left(\frac{c}{\sqrt{mn}}\right)^{2\sigma_1+2\sigma_2-7+\varepsilon_1}\right) \tag{2.40}$$

where $\sigma_1 = \operatorname{Re} s,\ \ \sigma_2 = \operatorname{Re}\rho$ and, for definiteness, we assume $\sigma_1 > \sigma_2$. Note $0 < \sigma_1 - 5/4,\ \sigma_2 - 5/4 < \varepsilon$ for some small $\varepsilon > 0$; under this condition the triple majorising series with terms (2.40) converges.

It means we can change the order of summation in (2.35).

Now we consider the inner sum of the Kloosterman sums. We would use identities (1.13) and (1.25) with the given h on the right side.

The reason of this manner is very simple: the conditions of my foretrace formulae with a given taste function in front of the Kloosterman sums are farther from necessary than the similar conditions for the case of the given taste function on the other side.

Proposition 2.4. *Let* $\operatorname{Re} s, \operatorname{Re}\rho \in (5/4, 5/4+\varepsilon)$ *for some small* $\varepsilon > 0$*; let* $h_0(r)$ *and* $g(k)$ *are defined by (2.19) and (2.21); then for any integers* $m, n \geqslant 1$ *we have for* $\operatorname{Re}\nu = \operatorname{Re}\mu = 1/2$

$$\sum_{c\geqslant 1} \frac{1}{c} S(n,m;c)\varphi_0\left(\frac{4\pi\sqrt{mn}}{c}\right) = \sum_{j\geqslant 1} \alpha_j t_j(n) t_j(m) h_0(\varkappa_j) +$$

$$+\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\tau_{1/2+ir}(n)\tau_{1/2+ir}(m)}{|\zeta(1/2+ir)|^2} h_0(r)\,dr + \sum_{k\geqslant 6} g(k) \sum_{1\leqslant j\leqslant 6} \alpha_{j,2k} t_{j,2k}(n) t_{j,2k}(m). \tag{2.41}$$

The function $h_0(r)$ coincides with the integral transform

$$h_0 = \pi\int_0^\infty k_0(x,1/2+ir)\varphi_0(x)\frac{dx}{x} =$$

$$= i\int_{(\Delta)} \gamma(s+\rho+w-1,1/2+ir)\gamma(1-s-w,\nu)\gamma(1-\rho-w,\mu)\cos\pi(s+\rho+w)\times$$

$$\times\Big(\cos\pi(s+w)\cos\pi(\rho+w)+\sin\pi\nu\sin\pi\mu\Big)\widehat{\Phi}_N(2w)\,dw \quad (2.42)$$

(if $|\operatorname{Im} r|$ be small and Δ be taken with condition $1 < \Delta + 2\sigma < 5/4$, $\sigma = \operatorname{Re} s = \operatorname{Re}\rho$, then we can integrate in (2.36) under the sign of integral over w).

One can see that $h_0(r)$ is regular in r at least for $|\operatorname{Im} r| < 2\sigma + \Delta - 1$ and we can take any Δ with $1 - \sigma - \Delta > 0$; the strip of regularity is the strip $|\operatorname{Im} r| < \sigma \approx 5/4$. If $r \to \infty$ inside of this strip we have $|h_0(r)| = O\left(|r|^{-5/2}\right)$ uniformly in $\operatorname{Im} r$, $|\operatorname{Im} r| < \sigma$. To find this estimate we move the path of integration to the left on the line $\operatorname{Re} w = \Delta_1$, $\Delta = -\sigma_1 - \sigma_2 + 1/4$ ($\approx -9/4$). The residue at $w = -s - \rho + 1 - ir$ is $O\left(|r|^{-N-5/2}\right)$ as it follows from the Stirling expansion; the same is true for the residue at $w = -s - \rho - ir$. On the line $\operatorname{Re} w = \Delta_1$ we have

$$\Big|\gamma(s+\rho+w-1,1/2+ir)\cos\pi(s+\rho+w)\Big| \ll$$

$$\ll \Big(\big||r|^2 - |w|^2\big| + 1\Big)^{\sigma_1+\sigma_2+\Delta_1-3/2}\exp\Big(\min\big(0,\pi(|w|-|r|)\big)\Big); \quad (2.43)$$

so the part of this integral with $|w| > |r|$ is exponentially small and it is sufficient consider the integral with $|w| \ll |r|$.

If $|w| \leqslant 2|r|$ then the integrand in (2.36) may be estimated as

$$\ll \Big(\big||r|^2 - |w|^2\big| + 1\Big)^{\sigma_1+\sigma_2+\Delta-3/2}\big(|w|+1\big)^{1-\sigma_1-\sigma_2-\Delta-N}. \quad (2.44)$$

Let $\Delta_1 = \Delta + \sigma_1 + \sigma_2$ ($\sigma_1 = \operatorname{Re} s, \sigma_2 = \operatorname{Re}\rho$); if $0 \leqslant \Delta_1 \leqslant 1/4$ then we have

$$\big|h_0(r)\big| \ll \int_0^{|r|/2} |r|^{2\Delta_1-3}(\eta+1)^{-\Delta_1-1-N}d\eta +$$

$$+\int_{|r|/2}^{|r|-1} |r|^{-5/2-N}\big(|r|-\eta\big)^{\Delta_1-3/2}d\eta + \int_{|r|-1}^{|r|+1} |r|^{-5/2-N}d\eta \ll |r|^{-5/2} \quad (2.45)$$

for any $N \geqslant 1$.

So all conditions of theorem 1.2 are fullfilled and first two terms on the right side (2.41) are equal to the sum of the Kloosterman sums with the taste function $\widetilde{\varphi}_0$,

$$\widetilde{\varphi}_0(x) = \int_{-\infty}^{\infty} k_0(x, 1/2 + ir) h_0(r)\, d\chi(r), \tag{2.46}$$

and with term which contains $\delta_{n,m}$.

One can see the integral (2.46) gives that component of φ_0 which is orthogonal to all Bessel's functions of the odd integer order. Let this component is $\varphi_0^{(H)}$; then

$$\varphi_0 = \varphi_0^{(H)} + \mathcal{N}, \tag{2.47}$$

where $\mathcal{N}$ is the corresponding Neumann series,

$$\mathcal{N}(x) = \sum_{k=1}^{\infty} 2(2k-1) J_{2k-1}(x) \int_0^{\infty} \varphi_0(y) J_{2k-1}(y) \frac{dy}{y} = \sum_{k=1}^{\infty} g(k) J_{2k-1}(x) \tag{2.48}$$

with $g(k)$ from (2.21) (again we can integrate term by term; after the calculation we change w by $w - s - \rho + 1$).

Taking in (2.21) $\Delta = -1 + \varepsilon$ with small $\varepsilon > 0$ (it is possible, since $\mathrm{Re}\,(w - s - \rho + 1) > -\frac{5}{2}$ on this line) we come to the estimate

$$|g(k)| \ll k^{-2+2\varepsilon}. \tag{2.49}$$

It rests use the Peterson identity and we get (2.41), since the singular term wich $\delta_{n,m}$ disappears (see [2]).

Proposition 2.5. *Under the same assumptions we have*

$$\sum_{c \geqslant 1} \frac{1}{c} S(n, -m; c) \varphi_1\left(\frac{4\pi\sqrt{nm}}{c}\right) = \sum_{j \geqslant 1} \alpha_j \varepsilon_j t_j(n) t_j(m) h_1(\varkappa_j) + \\ + \frac{1}{\pi} \int_{-\infty}^{\infty} \tau_{1/2+ir}(n) \tau_{1/2+ir}(m) h_1(r) \frac{dr}{|\zeta(1+2ir)|^2} \tag{2.50}$$

with h_1 from (2.20).

We define h_1 by the integral transform

$$h_1(r) \equiv h_1(r, s, \nu; \rho, \mu) = \pi \int_0^\infty \varphi_1(x) k_1(x, 1/2 + ir) \frac{dx}{x}, \tag{2.51}$$

where for $x > 0$

$$k_1(x, \nu) = \frac{2}{\pi} \sin \pi\nu K_{2\nu-1}(x) \tag{2.52}$$

The term by term integration gives for this function the representation (2.20). The function h_1 is satisfying to the same conditions as h_0; it is sufficient to have (2.50) if φ_1 be defined by the integral transform (1.26) of h_1. But this integral coincides with φ_1 if h_1 be defined by (2.51).

The union of all previous equalities and two last propositions gives us (2.21) if $\operatorname{Re} s,\ \operatorname{Re}\rho > 5/4$ and both differences $(\operatorname{Re} s - 5/4), (\operatorname{Re}\rho - 5/4)$ are sufficiently small.

2.3. The meromorphic continuation.

Let us introduce the following notation

$$\mathcal{R}(s, \nu; \rho, \mu | h) = 2\frac{\zeta(2s-1)\zeta(2\nu)}{\zeta(2-2s+2\nu)} \mathcal{Z}(\rho; \mu, 1-s+\nu) h\big(i(s-\nu-1/2)\big) \tag{2.53}$$

(it is the residue at the point $r = i(s-\nu-1/2)$ of the integrand in the definition $Z^{(c,+)}(s, \nu; \rho, \mu|h)$ with the additional factor $2\pi i$).

Proposition 2.6.. *Let h be a regular even function in the sufficiently wide strip $|\operatorname{Im} r| \leqslant \Delta,\ \Delta > 1/2$ and let $|h(r)| \ll |r|^{-B}$ for some $B > 5/2$ as $r \to \infty$ in this strip. Then for $1/2 \leqslant \operatorname{Re} s < 1, 1/2 \leqslant \operatorname{Re}\rho < 1$ the meromorphic continuation of $Z^{(c,+)}(s, \nu; \rho, \mu|h)$ is given by the equality*

$$\begin{aligned} Z^{(c,+)}(s, \nu; \rho, \mu|h) &= Z^{(c)}(s, \nu; \rho, \mu|h) + \\ &+ \mathcal{R}(s, \nu; \rho, \mu|h) + \mathcal{R}(s, 1-\nu; \rho, \mu|h) + \mathcal{R}(\rho, \mu; s, \nu|h) + \mathcal{R}(\rho, 1-\mu; s, \nu|h). \end{aligned} \tag{2.54}$$

Really,

$$Z^{(c,+)}(s, \nu; \rho, \mu|h) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathcal{Z}(s; \nu, 1/2+ir)\mathcal{Z}(\rho; \mu, 1/2+ir)}{\zeta(1+2ir)\zeta(1-2ir)} h(r)\, dr \tag{2.55}$$

and we have the Cauchy integral, because ζ has only a simple pole.

The product $\mathcal{Z}(s;\nu,1/2+ir)$ has the poles at the points $r_j,\ 1\leqslant j\leqslant 4$, with

$$ir_1=1/2-s+\nu,\quad ir_2=3/2-s-\nu,\quad ir_3=-r_1,\quad ir_4=-r_2.$$

When $\mathrm{Re}\, s>1$ the points r_1,r_2 are lying above the real axis and if $\mathrm{Re}\, s<1$, they are below the same. Now one can deform the path of integration (see picture; the deformation must be so small that the functions $\zeta(1\pm 2ir)$ have no zeros in side the lines; it is possible, since the Riemann zeta–function has no zeros on the line $\mathrm{Re}\, s=1$).

The path of integration l.

Firstly we fix $\rho,\ \ \mathrm{Re}\,\rho>1$; if $\mathrm{Re}\, s>1$ we have

$$Z^{(c,+)}(s,\nu;\rho,\mu|h)=\frac{1}{\pi}\int\limits_{(l)}(...)\,dr+\mathcal{R}(s,\nu;\rho,\mu|h)+\mathcal{R}(s,1-\nu;\rho,\mu|h); \tag{2.56}$$

it is the result of the direct calculation of the residues. But for $\mathrm{Re}\, s<1$ we have $\mathrm{Im}\, r_1,\ \mathrm{Im}\, r_2,<0$ and $\mathrm{Im}\, r_3,\ \mathrm{Im}\, r_4>0$, so we can integrate over the real axis, $\int\limits_l(\dots)\,dr=\int\limits_{-\infty}^{\infty}(\dots)\,dr$.

Doing the same with a fixed $s,\ \mathrm{Re}\, s<1$, we get the continuation in the variable ρ (for what it is sufficient to permute the pair of variables); it gives the equality (2.54).

As the consequence we can rewrite Theorem 2.2 and get our functional equation inside of the strip $1/2\leqslant \mathrm{Re}\, s,\ \mathrm{Re}\,\rho<1$.

Theorem 2.3. *Let $h(r)$ be the even regular function in the strip $|\mathrm{Im}|\, r\leqslant\Delta$ for some $\Delta>5/2,\ \ h(i/2)=h(3i/2)=0$ and $|h(r)|\ll|r|^{-B}$ for any fixed positive B when $r\to\infty$ inside of this strip.*

Let Φ_N be defined, for this given h, by (2.11) with $N \geqslant 1$ and $l_1 \geqslant 1$.

Then for any s, ν, ρ, μ with $\operatorname{Re}\nu = \operatorname{Re}\mu = 1/2$, $\nu, \mu \neq 1/2$ and $1/2 \leqslant \operatorname{Re} s, \operatorname{Re}\rho < 1$ we have

$$
\begin{aligned}
&Z^{(d)}(s,\nu;\rho,\mu|h,0) + Z^{(c)}(s,\nu;\rho,\mu|h,0) = \\
&\quad = Z^{(d)}(\rho,\nu;s,\mu|h_0,h_1) + Z^{(c)}(\rho,\nu;s,\mu|h_0+h_1) + Z^{(r)}(\rho,\nu;s,\mu|g) + \\
&\quad\quad + \frac{1}{2\pi}\sum_{l\in L}(-1)^l c(l) z_{2l}(s,\nu;\rho,\mu) + \\
&\quad + \mathcal{R}_h(s,\nu;\rho,\mu) + \mathcal{R}_h(s,1-\nu;\rho,\mu) + \mathcal{R}_h(\rho,\mu;s,\nu) + \mathcal{R}_h(\rho,1-\mu;s,\nu) - \\
&\quad - \Big(\mathcal{R}(s,\nu;\rho,\mu|h) + \mathcal{R}(s,1-\nu;\rho,\mu|h) + \mathcal{R}(\rho,\mu;s,\nu|h) + \mathcal{R}(\rho,1-\mu;s,\nu|h)\Big) + \\
&\quad + \mathcal{R}(\rho,\nu;s,\mu|h_0+h_1) + \mathcal{R}(\rho,1-\nu;s,\mu|h_0+h_1) + \\
&\quad + \mathcal{R}(s,\mu;\rho,\nu|h_0+h_1) + \mathcal{R}(s,1-\mu;\rho,\nu|h_0+h_1)
\end{aligned}
\tag{2.57}
$$

where h_0, h_1 and g are defined by (2.19)–(2.21) with $N \geqslant 1$, $l_1 \geqslant 1$ and $\mathcal{R}_h$, $\mathcal{R}$ are given by the equalities (2.18), (2.53).

§3. SPECIALIZATION OF THE MAIN FUNCTIONAL EQUATION

3.1. The choice of variables.

To estimate $|\zeta(1/2+it)|$ for large positive t we use the following specialization of the functional equation (2.57):

i)we take in this identity

$$s = \mu = 1/2, \nu = 1/2 + it, \rho = 1/2 + i\tau, \tag{3.1}$$

where t and τ are large, $t \to +\infty$ and $\tau \gg t^4$; we assume that for some fixed $\delta > 0$ the parameter t is larger than τ^δ and at the end we will take $t \approx \tau^{1/8}$).

ii)we assume that for all real r

$$h(r) > 0 \tag{3.2}$$

and this h satisfies to all conditions of Theorem 2.3;

iii) for the choosen h we suppose

$$\Phi(x) = \int_{-\infty}^{\infty} k_0(x, 1/2+iu) h(u)\, d\chi(u) - \sum_{l\in L} c(l) J_{2l-1}(x), \tag{3.3}$$

where L contains 5 elements

$$L = \{2, 3, 4, 5, 7\} \tag{3.4}$$

(these concrete elements are taken since the spaces of the corresponding cusp forms of the weight $2l$, $l \in L$, are empty; so sums z_{2l} on the right side (2.57) are zeroes).

The coefficients $c(l)$ in (3.3) are defined by equalities (2.10) with $N = 5$, so the Mellin transform of Φ is $O(|w|^{2\mathrm{Re}w-6})$ if $w \to \infty$ and $\mathrm{Re}w$ is fixed.

For our specialzation we have on the left side (2.57) the very fast convergent series

$$\sum_{j\geqslant 1} \alpha_j |\mathcal{H}_j(\nu)|^2 \mathcal{H}_j^2(\rho) h(\varkappa_j) \tag{3.5}$$

and the similar integral over the continuous spectrum.

3.2. The averaging. Let T_0 be sufficiently large and $T = T_0^{1-\varepsilon}$ with small fixed $\varepsilon > 0$. We suppose

$$\omega_T(\rho) = \frac{1}{4}\Big(\cos\frac{\pi}{2T}(\rho - 1/2 - iT_0)\Big)^{-1} \tag{3.6}$$

and consider the average of our functional equation (2.57) with variables (3.1) over variable ρ with weight $\omega_T(\rho)$.

The average of $Z^{(d)}$ is the integral

$$\frac{1}{i}\int_{(1/2)} \omega_T(\rho) \sum_{j\geqslant 1} \alpha_j |\mathcal{H}_j(\nu)|^2 \mathcal{H}_j^2(\rho) h(\varkappa_j)\, d\rho \tag{3.7}$$

where we can integrate term by term under our conditions for h.

All integrals

$$\frac{1}{i}\int_{(1/2)} \omega_T(\rho) \mathcal{H}_j^2(\rho)\, d\rho \tag{3.8}$$

may be calculated over the line $\mathrm{Re}\rho = 2$, where $\mathcal{H}_j$ is the absolutely convergent series and we have

$$\frac{1}{i}\int_{(1/2)} \omega_T(\rho) \mathcal{H}_j^2(\rho)\, d\rho = T \sum_{n,m\geqslant 1} \frac{t_j(n)t_j(m)}{(nm)^{1/2+iT_0}} ((nm)^T + (nm)^{-T})^{-1} \tag{3.9}$$

(we use the so called Ramanujan integral here).

The series on the riht side (3.9) equals to

$$T\left(1+O(2^{-T})\right) \tag{3.10}$$

and the main term is positive.

Doing the same in the integral over the continuous spectrum we come to the following expression for the average on the left side

$$\frac{1}{i}\int\limits_{(1/2)} \omega_T(\rho)\{Z^{(d)}(1/2,\nu;\rho,1/2|h,0)+Z^{(c)}(1/2,\nu;\rho,1/2|h,0)\}\,d\rho =$$

$$= T\{\sum_{j\geqslant 1}\alpha_j|\mathcal{H}_j(\nu)|^2 h(\varkappa_j)+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\zeta(\nu+ir)\zeta(\nu-ir)|^2}{|\zeta(1+2ir)|^2}h(r)\,dr\}+o(1) \tag{3.11}$$

Really, we can integrate over ρ under the sign of integration in r. When we move the path of integration to the line $\operatorname{Re}\rho=2$ the additional terms will be appeared from the poles of $\zeta^2(\rho+ir)\zeta^2(\rho-ir)$. But the products $h(r)\omega_T(1\pm ir)$, $h(r)\omega_T'(1\pm ir)$ are $O(T^{-M})$ for any positive M.

The upper bound for this expression will be received by the same averaging on the right side.

Here we have the products $\mathcal{H}_j(\rho+\nu-1/2)\mathcal{H}_j(\rho-\nu+1/2)\mathcal{H}_j^2(1/2)$ in the sum over the discret spectrum and $\mathcal{Z}(\rho;\nu,1/2+ir)\mathcal{Z}(1/2;1/2,1/2+ir)$ in the integral, but coefficients h_0 and h_1 are depending in ρ also and we must receive the explicit formulas for them.

For this purpose we will express these coefficients in terms of the integrals with two hypergeometric functions.

For large values τ, t and r we can write the full asymptotic expansions for these hypergeomtric functions. It allows us receive the sufficiently exact estimates for the average on the right side.

As the result we have the upper bound for (3.11) in the form

$$\sum_{\varkappa_j\leqslant\sqrt{T_0}}\alpha_j\mathcal{H}_j^2(1/2)+\sum_{\sqrt{T_0}<\varkappa_j\leqslant 2T_0}\alpha_j\frac{\log^2\varkappa_j}{\varkappa_j^2}\mathcal{H}_j^2(1/2)+O(T_0^{1+\varepsilon}) \tag{3.12}$$

plus the similar sum over regular cusp forms (see § 9).

These mean values are known and it gives the estimate $O(T^{1+\varepsilon})$ for (3.11); of course, it is sufficient to prove (0.2) and (0.3).

§ 4.NEW REPRESENTATIONS FOR THE COEFFICIENTS

4.1. The definition of the kernels A_{jk}. The Mellin integrals (2.19)–(2.21) are very convenient in general theory, but these ones are not good for the special cases when one (ore more) variable is large. There is other form which is more suitable for the estimation in these cases.

We define four kernels $A_{jk}(r,x;s,\nu)$, $\;0\leqslant j,k\leqslant 1$, by the equalities

$$(4\pi)^{2s-1}A_{0j}(r,x;s,\nu)=\pi x\int\limits_0^\infty k_0(y,1/2+ir)k_j(xy,\nu)y^{2s-1}dy,\quad j=0,1; \tag{4.1}$$

$$(4\pi)^{2s-1}A_{1j}(r,x;s,\nu)=\pi x\int\limits_0^\infty k_1(y,1/2+ir)k_{1-j}(xy,\nu)y^{2s-1}dy,\quad j=0,1; \tag{4.2}$$

here $k_0(x,\nu)$ and $k_1(x,\nu)$ are defined by (1.17) and (2.51) correspondingly.

The integral for A_{00} is absolutely convergent in the strip $1/2>\mathrm{Re}\,s>|\mathrm{Im}\,r|+|\mathrm{Re}\,(\nu-1/2)|$ and all others are convergent in the halfplane $\mathrm{Re}\,s>|\mathrm{Im}\,r|+|\mathrm{Re}\,(\nu-1/2)|$.

Note the kernels A_{0j} have a sense for $r=\pm i(l-1/2)$ with an integer l since

$$k_0(y,l)=k_0(y,1-l)=(-1)^lJ_{2l-1}(y)$$

and the corresponding integrals (4.1) with $l=\pm i(l-1/2)$ are convergent for $1/2>\mathrm{Re}s>-l+1/2+|\mathrm{Re}(\nu-1/2)|$.

4.2. The explicit formulas. Here (and for what will be later on) it is convenient rewrite the explicit formulas for A_{jk} from [8] (with the additional representations which are the immediate consequence of the Kummer relations between the Gauss hypergeometric functions of arguments x, $1-x$, $-\frac{1}{x}$, ...).

It is very essentially for us that the function

$$\mathcal{F}(x;s,\nu,r)=|x|^{ir+1/2}\,(1-x)^s\,F(s+\nu-1/2+ir,s-\nu+1/2+ir;1+2ir;\,x) \tag{4.3}$$

(here the usual notation for the Gauss hypergeometric function is used; it is assumed that x be real and $x<1$) is a solution of the differential equation

$$\frac{d^2\mathcal{F}}{dx^2}+\left(\frac{r^2}{x^2\,(1-x)}-\frac{(s-1/2)^2}{x\,(1-x)^2}+\frac{(\nu-1/2)^2}{x\,(1-x)})+\frac{1-x+x^2}{4x^2\,(1-x)^2}\right)\mathcal{F}=0. \tag{4.4}$$

It gives the possibility find the asymptotic expansions for $\mathcal{F}$ if at least one parameter be large (see §5).

The first assertion about A_{jk} follows directly from the definition.

Proposition 4.1. *Let A_{jk} are defined by two integrals (4.1) and (4.2). Then these functions for any fixed positive x and for any fixed ν with* $\mathrm{Re}\nu = 1/2$ *are regular in s and r if* $\mathrm{Re}s > |\mathrm{Im}r|$.

The formulas for A_{jk} below give us the meromorphic continuation of these functions; the result of this continuation we will denote by the same symbol.

Proposition 4.2. *Let us introduce the notations*

$$a = s + \nu - 1/2 + ir,\ b = s - \nu + 1/2 + ir,\ c = 1 + 2ir, \tag{4.5}$$

$$a' = s + \nu - 1/2 + ir,\ b' = s - \nu + 1/2 + ir,\ c' = 1 - 2ir. \tag{4.6}$$

With these notations for $0 \leqslant x < 1$ we have

$$\begin{aligned}(2\pi)^{2s-1}\, A_{00}(r, x; s, \nu) &= \\ = \frac{1}{2\cos\pi\nu}\Bigg(\sin\pi(s+\nu)&\frac{\Gamma(a)\Gamma(a')}{\Gamma(2\nu)}\, x^{2\nu}\, F(a, a'; c;\, x^2) + \\ &+ \sin\pi(s-\nu)\frac{\Gamma(b)\Gamma(b')}{\Gamma(2-2\nu)}\, x^{2-2\nu}\, F(b, b'; 2-2\nu; x^2)\Bigg)\end{aligned} \tag{4.7}$$

Proposition 4.3. *Let $x > 1$; then we have*

$$\begin{aligned}(2\pi)^{2s-1} A_{00}(r, x; s, \nu) = \frac{ix^{1-2s}}{2\sinh\pi r}\Bigg(&\cos\pi(s+ir)\frac{\Gamma(a)\Gamma(b)}{x^{c-1}\Gamma(c)} F(a, b; c; \frac{1}{x^2}) - \\ &- \cos\pi(s-ir)\frac{\Gamma(a')\Gamma(b')}{x^{c'-1}\Gamma(c')} F(a', b'; c'; \frac{1}{x^2})\Bigg)\end{aligned} \tag{4.8}$$

The next representation gives the union of the previous ones.

Proposition 4.4. *For all $x \geqslant 0$*

$$\begin{aligned}(2\pi)^{2s-1} A_{00}(r, x; s, \nu) &= \sin\pi s\, \Gamma(2s-1)|x^2-1|^{1-2s} x^{2\nu} F(1-b, 1-b'; 2-2s; 1-x^2) \\ &+ \frac{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')}{2\pi\cos\pi s\, \Gamma(2s)}(\cosh^2\pi r + \sin^2\pi\nu - \sin^2\pi s) x^{2\nu} F(a, a'; 2s; 1-x^2)\end{aligned} \tag{4.9}$$

Two kernels A_{01} and A_{10} are exponentially small when $r \to \pm\infty$ (and s, ν are $o(|r|)$.

Proposition 4.5. *For all $x > 0$ we have*

$$(2\pi)^{2s-1}A_{01}(r,x;s,\nu) = \frac{ix^{1-2s}\sin\pi\nu}{2\sinh\pi r}\left(\frac{\Gamma(a)\Gamma(b)}{x^{c-1}\Gamma(c)}F(a,b;c;-\frac{1}{x^2})\right.$$
$$\left. - \frac{\Gamma(a')\Gamma(b')}{x^{c'-1}\Gamma(c')}F(a',b';c';-\frac{1}{x^2}\right) \quad (4.10)$$

and the same kernel equals to

$$\frac{1}{2\cos\pi\nu}\left(\sin\pi(s+\nu)\frac{\Gamma(a)\Gamma(a')}{\Gamma(2\nu)}x^{2\nu}F(a,a';2\nu;-x^2)+\right.$$
$$\left. + \sin\pi(s-\nu)\frac{\Gamma(b)\Gamma(b')}{\Gamma(2-2\nu)}x^{2-2\nu}F(b,b';2-2\nu;-x^2)\right). \quad (4.11)$$

Proposition 4.6. *For all $x \geqslant 0$ we have*

$$(2\pi)^{2s-1}A_{10}(r,x;s,\nu) = \cosh\pi r\sin\pi\nu\frac{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')}{\pi\Gamma(2s)}x^{2\nu}F(a,a';2s;1-x^2); \quad (4.12)$$

at the same time this kernel for $0 \leqslant x < 1$ equals to

$$\frac{\cosh\pi r}{2\cos\pi\nu}\left(\frac{\Gamma(a)\Gamma(a')}{\Gamma(2\nu)}x^{2\nu}F(a,a';2\nu;x^2) - \frac{\Gamma(b)\Gamma(b')}{\Gamma(2-2\nu)}x^{2-2\nu}F(b.b';2-2\nu;x^2)\right) \quad (4.13)$$

and

$$\frac{ix^{1-2s}\sin\pi\nu}{2\sinh\pi r}\left(\frac{\Gamma(a)\Gamma(b)}{x^{c-1}\Gamma(c)}F(a,b;c;\frac{1}{x^2}) - \frac{\Gamma(a')\Gamma(b')}{x^{c'-1}\Gamma(c')}F(a',b';c';\frac{1}{x^2})\right) \quad (4.14)$$

if $x > 1$.

Finally, for the last kernel we have

Proposition 4.7. *Let A_{11} be defined by (4.2); then for all $x > 0$*

$$(2\pi)^{2s-1}A_{11}(r,x;s.\nu) = \frac{ix^{1-2s}}{2\sinh\pi r}\left(\cos\pi(s+ir)\frac{\Gamma(a)\Gamma(b)}{x^{c-1}\Gamma(c)}F(a,b;c;-\frac{1}{x^2})-\right.$$
$$\left. - \cos\pi(s-ir)\frac{\Gamma(a')\Gamma(b')}{x^{c'-1}\Gamma(c')}F(a',b';c';-\frac{1}{x^2})\right) \quad (4.15)$$

and this kernel equals to

$$\frac{\cosh \pi r}{2\cos \pi\nu}\left(\frac{\Gamma(a)\Gamma(a')}{\Gamma(2\nu)}x^{2\nu}F(a,a';2\nu;-x^2)- \right.$$
$$\left. -\frac{\Gamma(b)\Gamma(b')}{\Gamma(2-2\nu)}x^{2-2\nu}F(b,b';2-2\nu;-x^2)\right) \quad (4.16)$$

4.3.The new representations for h_0 and h_1. In this subsection we express the integrals (2.19) and (2.20) in terms of integrals with the kernels A_{jk}; it gives us the method to estimate these coefficients.

Lemma 4.1. *Let*

$$0 \leqslant |\mathrm{Im} r| < 1/2 \leqslant \mathrm{Re} s, \mathrm{Re}\rho < 1, \nu = \mu = 1/2$$

and let the function $h(r)$ be taken under all conditions of Theorem 2.3.

Then we have the following equalities for two integrals (2.19) and (2.20):

$$h_j = \frac{1}{2\pi}\int_{-\infty}^{\infty} B_j(r,u;s,\nu;\rho,\mu)\,h(u)\,d\chi(u) - \sum_{l\in L}(-1)^l c(l) b_{j,l}(s,\nu;\rho,\mu) \quad ((4.17))$$

where for $j=0$ and $j=1$

$$B_j(r,u;s,\nu;\rho,\mu) = B_{j0}(r,u;s,\nu;\rho,\mu) + B_{j1}(r,u;s,\nu;\rho,\mu) \quad ((4.18))$$

with

$$B_{0j} = \int_0^{\infty} A_{0j}(r,\sqrt{x};\rho,\nu)\,A_{0j}(u,\frac{1}{\sqrt{x}};1-\rho,\mu)\,x^{\rho-s-1}\,dx, \quad (4.19)$$

$$B_{1j} = \int_0^{\infty} A_{1j}(r,\sqrt{x};\rho,\nu)\,A_{0j}(u,\frac{1}{\sqrt{x}};1-\rho,\mu)\,x^{\rho-s-1}\,dx \quad (4.20)$$

and

$$b_{jl}(r;s,\nu;\rho,\mu) = B_j(r,-(l-1/2)i;s,\nu;\rho,\mu). \quad (4.21)$$

Furthermore, there is the symmetry

$$B_j(r,u;s,\nu;\rho,\mu) = B_j(r,u;\rho,\mu;s,\nu) \quad (4.22)$$

(the pair (s,ν) have been replaced by (ρ,μ) and otherwise).

The proof is based on the known Mellin's transforms for the Bessel functions:

$$\int_0^\infty k_0(x,\nu)\,x^{2w-1}\,dx = \gamma(w,\nu)\cos\pi w, \;\; |\mathrm{Re}\nu - 1/2| < \mathrm{Re}w < 1/2, \tag{4.23}$$

$$\int_0^\infty k_1(x,\nu)\,x^{2w-1}\,dx = \gamma(w,\nu)\sin\pi\nu, \;\; |\mathrm{Re}\nu - 1/2| < \mathrm{Re}w \tag{4.24}$$

(the combinations of the Bessel functions k_0 and k_1 are defined by (1.17) and (2.51), $\gamma(w,u)$ is given by (1.31)).

We substitute the definition of $\widehat{\Phi}$ and change the order of the integration; it is possible since we assume the fast decreasing of h.

We will use the folloing fact from the Mellin theory. We say that $f_1, f_2 \in L^2$ if

$$\int_0^\infty |f_j|^2\,\frac{dx}{x} < \infty. \tag{4.25}$$

If for some Δ we have $x^{\Delta} f_1 \in L^2, x^{\sigma-\Delta} f_2 \in L^2$, then for any s with the condition $\mathrm{Re}s = \sigma$ we have

$$\int_0^\infty f_1\,f_2\,x^{s-1}\,dx = \frac{1}{2\pi i}\int\limits_{(\Delta)} F_1(w)\,F_2(s-w)\,dw, \tag{4.26}$$

where the integration is doing over the line $\mathrm{Re}\,w = \Delta$ and F_j are the Mellin transform f_j.

4.3.1. Function B_{00}

Two functions

$$f_1(y) = k_0(y, 1/2+ir)y^{2c}, f_2(y) = k_0(xy,\nu)y^{2s-2c-1} \tag{4.27}$$

for any fixed $x > 0$ satisfy to condition (4.25) if for s with $\mathrm{Re}s > 1/2$ the parameter c be taken with condition

$$\max(0, \sigma - 3/4) < \mathrm{Re}c < \min(1/4, \sigma - 1/2), \sigma = \mathrm{Re}s.$$

Using now (4.23) we come to the equality

$$\frac{(4\pi)^{2s-1}}{\pi\sqrt{x}} A_{00}(r,\sqrt{x};s,\mu) = $$
$$= \frac{1}{2\pi i}\int\limits_{(0)} \gamma(c+w/2, 1/2+ir)\gamma(s-c-w/2,\mu)\cos\pi(c+w/2)\times$$
$$\times \cos\pi(s-c-w/2) x^{-s+c+w/2}\, dw$$
$$= -\frac{1}{\pi i}\int\limits_{(\Delta)} \gamma(1-w, 1/2+ir)\gamma(s-1+w,\mu)\cos\pi w\cos\pi(s+w)\times$$
$$\times x^{1-s-w}\, dw \quad (4.28)$$

(here we can take any Δ with $1>\Delta>1-\mathrm{Re}s$).

This equality means that two functions

$$2(4\pi)^{2s-2}\, x^{s-3/2}\, A_{00}(r,\sqrt{x};s,\mu)$$

and

$$F_1(w) = -\gamma(1-w, 1/2+ir)\gamma(s-1+w,\mu)\cos\pi w\cos\pi(s+w)$$

are the Mellin pair.

By the same way we come to the second pair

$$2(4\pi)^{-2s}\, x^{1/2-\rho}\, A_{00}(u, \frac{1}{\sqrt{x}}; 1-s,\nu)$$

and

$$F_2(w) = -\gamma(w+1-s-\rho, 1/2+iu)\gamma(\rho-w,\nu)\cos\pi(\rho-w)\cos\pi(w-s-\rho)$$

(in (4.28) we change μ, r by ν, u and take $c=1-s-\rho$).

Using (4.26) again we come to the equality

$$\frac{1}{2\pi i}\int F_2(w)\, F_1(1-w)\, dw =$$
$$= \frac{1}{2\pi i}\int\limits_{(\Delta)} \gamma(w, 1/2+ir)\gamma(s-w,\mu)\gamma(w+1-s-\rho, 1/2+iu)\gamma(\rho-w,\nu)\times$$
$$\times \cos\pi w\cos\pi(s-w)\cos\pi(\rho-w)\cos\pi(w-s-\rho)\, dw$$
$$= \frac{1}{4\pi^2}\int_0^\infty A_{00}(r,\sqrt{x};s,\mu) A_{00}(u,\frac{1}{\sqrt{x}};1-s,\nu)\, x^{s-\rho-1}\, dx \quad (4.29)$$

(here $\max(0, \mathrm{Re}(s+\rho)-1) < \Delta < \min(\mathrm{Re}s, \mathrm{Re}\rho)$).

If we take (4.28) with ρ, ν instead of s, μ and the second pair with ρ, μ instead of s, ν then we receive the equality

$$\frac{1}{2\pi i}\int\limits_{(\Delta)} \gamma(w, 1/2+ir)\gamma(\rho-w, \nu)\gamma(w+1-s-\rho, 1/2+iu)\gamma(s-w, \mu)\times$$
$$\times \cos\pi w \cos\pi(\rho-w)\cos\pi(s-w)\cos\pi(w+1-s-\rho)\, dw =$$
$$= \frac{1}{4\pi^2}\int_0^\infty A_{00}(r, \sqrt{x}; \rho, \nu)\, A_{00}(u, \frac{1}{\sqrt{x}}; 1-s, \nu)\, x^{\rho-s-1}\, dx \quad (4.30)$$

It gives us the equality (4.19) for $j=0$ and we see that

$$B_{00}(r, u; s, \nu; \rho, \mu) = B_{00}(r, u; \rho, \mu; s, \nu). \quad (4.31)$$

4.3.2. Function B_{01}

Using the same formulas we have

$$2(4\pi)^{2\rho-2}\int_0^\infty x^{\rho-3/2} A_{01}(r, \sqrt{x}; \rho, \nu)\, x^{-w}\, dw =$$
$$= \gamma(w, 1/2+ir)\gamma(\rho-w, \nu)\cos\pi w \sin\pi\nu \quad (4.32)$$

and

$$2(4\pi)^{-2\rho}\int_0^\infty x^{1/2-s}\, A_{01}(u, \frac{1}{\sqrt{x}}; 1-\rho, \mu)\, x^{w-1}\, dw =$$
$$= -\gamma(s-w, \mu)\gamma(w+1-s-\rho, 1/2+iu)\sin\pi\mu\cos\pi(w-s-\rho) \quad (4.33)$$

As the consequence we have the equality (4.19) for the case $j=1$; after the simultaneous change the pair (ρ, ν) by (s, μ) in (4.32) and (4.33) and taking into account (4.31) we come to the equality (4.22) for the case $j=0$.

4.3.3. Functions B_{10} and B_{11}

First of all we have two equalities with $\gamma(w, 1/2+ir)$:

$$2(4\pi)^{2s-2}\int_0^\infty x^{s-3/2} A_{11}(r, \sqrt{x}; s, \mu)\, x^{-w}\, dw =$$
$$= \gamma(w, 1/2+ir)\gamma(s-w, \mu)\cosh\pi r\cos\pi(s-w), \quad (4.34)$$

$$2(4\pi)^{2\rho-2}\int_0^\infty x^{\rho-3/2}A_{10}(\rho,\sqrt{x};\rho,\nu)\,x^{-w}\,dx=\gamma(w,1/2+ir)\gamma(\rho-w,\nu)\times \\ \times\cosh\pi r\sin\pi\nu \quad (4.35)$$

Furthermore,

$$2(4\pi)^{-2s}\int_0^\infty x^{1/2-\rho}A_{01}(u,\frac{1}{\sqrt{x}};1-s,\nu)\,x^{w-1}\,dx= \\ =-\gamma(w-s-\rho+1,1/2+iu)\gamma(\rho-w,\nu)\sin\pi\nu\cos\pi(w-s-\rho), \quad (4.36)$$

$$2(4\pi)^{-2\rho}\int_0^\infty x^{1/2-s}A_{00}(u,\frac{1}{\sqrt{x}};1-\rho,\mu)\,x^{w-1}\,dx= \\ =-\gamma(w-s-\rho+1,1/2+iu)\gamma(s-w,\mu)\cos\pi(s-w)\cos\pi(w-s-\rho). \quad (4.37)$$

It follows from these equalities that we have two representations

$$i\int_{(\Delta)}\gamma(w,1/2+ir)\gamma(\rho-w,\nu)\gamma(s-w,\mu)\gamma(w-s-\rho+1)\times \\ \times\cosh\pi r\sin\pi\nu\cos\pi(s-\nu)\cos\pi(w-s-\rho)\,dw= \\ =\frac{1}{2\pi}\int_0^\infty A_{11}(r,\sqrt{x};s,\mu)\,A_{01}(u,\frac{1}{\sqrt{x}};1-s,\nu)\,x^{s-\rho-1}\,dx \\ =\frac{1}{2\pi}\int_0^\infty A_{10}(r,\sqrt{x};\rho,\nu)A_{00}(u,\frac{1}{\sqrt{x}};1-\rho,\mu)\,x^{\rho-s-1}\,dx \quad (4.38)$$

Replacing here the pair (s,ν) by (ρ,μ) (and writing (s,ν) instead of (ρ,μ) we come to the equality

$$B_{10}(r,u;s,\nu;\rho,\mu)=B_{11}(r,u;\rho,\mu;s,\nu) \quad (4.39)$$

and we have (4.22) for the case $j=1$ also.

All integrals with the Bessel function J_{2l-1} (instead of J_{2iu}) have been considered by the same way and we can omitt the corresponding computations.

§5. ASYMPTOTIC FORMULAS FOR THE KERNELS

It is necessary now write the asymptotic expansions for the Gauss hypergeometric functions in the integral representations for h_j when $s = \mu = 1/2$, $\nu = 1/2 + it$ and positive t and r are sufficintly large.These formulas are given in subsections 5.1 – 5.5.

Of course, these formulas must be taken from standard handbooks, but nobody wrote these ones. The methods of the asymptotic integration of the ordinary differential equations of the second order with a large parameter are well known (see, for example, [9]). For our kernels it rests determine the corresponding normalizing coefficients from the boundary conditions to write the asymptotic expansion.

5.1.The kernel $A_{01}(u, x; 1/2, \nu)$**.**

Let $\nu = 1/2 + it$ and $t \to +\infty$. We define two asymptotic series

$$\mathcal{A}(\xi, u; t) = \sum_{n\geqslant 0} \frac{a_n(\xi, u)}{t^{2n}},\ \mathcal{B}(\xi, u; t) = \sum_{n\geqslant 1} \frac{b_n(\xi, u)}{t^{2n}}$$

by the following recurrent relations: $a_0 \equiv 1$, $b_0 \equiv 0$ and for $n \geqslant 0$

$$b'_{n+1} = \frac{1}{2}(a''_n - f a_n) - (4u^2 + 1/4)\xi^{-1}(\xi^{-1} b_n)', \tag{5.1}$$

$$a'_n = \frac{1}{2}(-b''_n + f b_n), \tag{5.2}$$

where $' = \frac{d}{d\xi}$ and

$$f = u^2\left(\frac{4}{\xi^2} - \frac{1}{\sinh^2 \xi/2}\right) + \frac{1}{4}\left(\frac{1}{\xi^2} - \frac{1}{\sinh^2 \xi}\right). \tag{5.4}$$

To integrate (5.1) and (5.2) we take the initial conditions

$$b_n(0) = 0, a_n(0) = -(1/2 + 2iu)b'_n(0); \tag{5.5}$$

note that all a_n are even and all b_n are odd functions in ξ and for $|\xi| < \pi$ they may be represented by the convergent power series.

After determination a_n, b_n from the recurrent relations we define for an integer $M \geqslant 1$ the following segment of the asymptotic series:

$$V_M(\xi; u) = \sqrt{\xi} J_{2iu}(t\xi) \sum_{0\leqslant n\leqslant M} \frac{a_n}{t^{2n}} + (\sqrt{\xi} J_{2iu}(t\xi))' \sum_{1\leqslant n\leqslant M} \frac{b_n}{t^{2n}} \tag{5.6}$$

With this notations we have the following uniform expansion.

Lemma 5.1. *Let $t \to +\infty$; then for real $u, |u| \ll t^{\varepsilon}$ for some small fixed $\varepsilon > 0$ we have uniformly in $\xi \geqslant 0$ for any fixed integer $M \geqslant 1$*

$$\sqrt{\sinh(\xi)}A_{01}(u, \frac{1}{\sinh\xi/2}; 1/2, 1/2+it) =$$

$$= \frac{i}{2\sinh \pi u}(\lambda(u,t)V_M(\xi;u) - \lambda(-u,t)V_M(\xi,-u) + O(\min(\sqrt{\xi}, \frac{1}{\sqrt{t}})t^{-2M-2})), \tag{5.7}$$

where

$$\lambda(u,t) = t^{-2iu}\Gamma(1/2+it+iu)\Gamma(1/2-it+iu)\cosh \pi t. \tag{5.8}$$

This assertion is the consequence of two facts.

The first one follows from (4.4): function

$$V = \sqrt{\sinh\xi}A_{01}(u, (\sinh\xi/2)^{-1}; 1/2, 1/2+it)$$

satisfies to the differential equation

$$V'' + (t^2 + \frac{u^2}{\sinh^2\xi/2} + \frac{1}{4\sinh^2\xi})V = 0 \tag{5.9}$$

or, the same,

$$V'' + (t^2 + \frac{4u^2}{\xi^2} + \frac{1}{4\xi^2})V = fV \tag{5.10}$$

with f from (5.4).

The second fact follows from (4.10): we have

$$A_{01}(u, \frac{1}{\sinh\xi/2}; 1/2, 1/2+it) =$$

$$= \frac{i}{2\sinh\pi u}\left(\frac{\lambda(u,t)}{\Gamma(1+2iu)}(\frac{t\xi}{2})^{2iu}(1+O(\xi^2)) - \right.$$

$$\left. - \frac{\lambda(-u,t)}{\Gamma(1-2iu)}(\frac{t\xi}{2})^{-2iu}(1+O(\xi^2))\right) \tag{5.11}$$

as $\xi \to 0$.

The uniform nearness of the solutions of the equation (5.10) to the corresponding Bessel functions is the well known fact.

So it rests check the coincidence of the asymptotic expansions on the right sides when $\xi \to 0$. But the initial values a_n and b_n have been taken from this condition namely.

5.2. The kernels $A_{01}((l-1/2)i, x; 1/2, \nu)$

For all integers $l \geqslant 1$ we have

$$A_{01}(l-1/2)i, \frac{1}{\sinh \xi/2}; 1/2, 1/2+it) =$$
$$= (-1)^l \cosh \pi t \frac{\Gamma(l+it)\Gamma(l-it)}{\Gamma(2l)} (\sinh \xi/2)^{2l-1} F(l+it, l-it; 2l; -\sinh^2 \xi/2); \quad (5.12)$$

so when $\xi \to 0$ this function equals to

$$(-1)^l \frac{\lambda(-(l-1/2)i, t)}{\Gamma(2l)} (\frac{t\xi}{2})^{2l-1} (1+O(\xi^2)) \quad (5.13)$$

It means we must take that solution of equation (5.10) which is near to the function $const \times \sqrt{\xi} J_{2l-1}(t\xi)$. More exactly, we have

Lemma 5.2. *Let t be sufficiently large and a_n, b_n are defined by the recurrent relations (5.1)–(5.5) with $u = (l-1/2)i$. Then for any fixed integer $l \geqslant 1$ we have (uniformly in $\xi \geqslant 0$ and for any fixed $M \geqslant 1$)*

$$\sqrt{\sinh} \xi A_{01}(i(l-1/2), \frac{1}{\sinh \xi/2}; 1/2, 1/2+it) =$$
$$= (-1)^l \lambda(-i(l-1/2), t) \Bigg(\sqrt{\xi} J_{2l-1}(t\xi) \sum_{0 \leqslant n \leqslant M} \frac{a_n}{t^{2n}}$$
$$+ (\sqrt{\xi} J_{2l-1}(t\xi))' \sum_{1 \leqslant n \leqslant M} \frac{b_n}{t^{2n}} + O(\min(\sqrt{\xi}(t\xi)^{2l-1}, \frac{1}{\sqrt{t}}) t^{-2M-2}) \Bigg) \quad (5.14)$$

5.3. The kernel $A_{11}(r, x; 1/2, 1/2)$

We define the sequence $\{c_n, d_n\}$ by the following recurrent relations: $c_0 \equiv 1, d_0 \equiv 0$ and for $n \geqslant 0$

$$d'_{n+1} = \frac{1}{2}(c_n - \frac{1}{2\xi}(\frac{d_n}{\xi})' - f_1 c_n), \quad (5.15)$$

$$c'_n = \frac{1}{2}(-d''_n + f_1\, d_n), \tag{5.16}$$

where $' = \frac{d}{d\xi}$ and

$$f_1 = \frac{1}{4}(\frac{1}{\xi^2} - \frac{1}{\sinh^2 \xi}) \tag{5.17}$$

We suppose for all $n \geqslant 1$

$$d_n(0) = 0 \tag{5.18}$$

(so $d_n(\xi)$ is an odd function in ξ and $(\xi^{-1} d_n(\xi))' = \frac{1}{3}d'''(0) + ...$ as $\xi \to 0$; for this reason $\xi^{-1}(\xi^{-1}d_n(\xi))'$ is finite in the neighbourhood of the point $\xi = 0$).

We take the boundary values for c_n in the form

$$c_n(0) = -\frac{1}{2}d'_n(0),\ n \geqslant 1, \tag{5.19}$$

so we can write instead of (5.15) and (5.16):

$$d_{n+1}(\xi) = \frac{1}{2}c'_n(\xi) - \frac{1}{4}\int_0^\xi (\frac{d_n(\eta)}{\eta})' \frac{d\eta}{\eta} - \frac{1}{2}\int_0^\xi f_1(\eta) c_n(\eta)\, d\eta,\ n \geqslant 0 \tag{5.20}$$

$$c_n(\xi) = -\frac{1}{2}d'_n(\xi) + \frac{1}{2}\int_0^\xi f_1(\eta) d_n(\eta)\, d\eta, n \geqslant 1. \tag{5.21}$$

With these functions we define the asymptotic expansion

$$V_{11}(r, \xi) = \sqrt{\xi} Y_0(r\xi) \sum_{n \geqslant 0} \frac{c_n}{r^{2n}} + (\sqrt{\xi} Y_0(r\xi))' \sum_{n \geqslant 1} \frac{d_n}{r^{2n}} \tag{5.22}$$

and let $V_{11,M}$ denotes sum of the first terms with $n \leqslant M$.

The function $(2 \tanh \xi/2)^{-\frac{1}{2}} A_{11}(r, \sinh \xi/2; 1/2, 1/2)$ is the solution of the differential equation (again this fact is the consequence of (4.4))

$$V'' + (r^2 + \frac{1}{4\sinh^2 \xi})V = 0 \tag{5.22}$$

or, the same,

$$V'' + (r^2 + \frac{1}{4\xi^2})V = f_1(\xi)\, V. \tag{5.23}$$

The function V_{11} is the asymptotic solution of this equation (the recurrent relations (5.20)–(5.21) are taken from this condition).

It is possible that these two solutions are proportional; but for our purposes the more weak assertion will be sufficient.

We define the second asymptotic solution, supposing

$$\widetilde{V}_{11} = \sqrt{\xi} J_0(r\xi) \sum_{n\geqslant 0} \frac{\tilde{c}_n}{r^{2n}} + (\sqrt{\xi} J_0(r\xi))' \sum_{n\geqslant 1} \frac{\tilde{d}_n}{r^{2n}}, \tag{5.25}$$

where $\tilde{c}_0 \equiv 1$ and $\tilde{c}_n, \tilde{d}_n$ for $n \geqslant 1$ are defined by the same recurrent relations (5.15)–(5.16) but with the following initial conditions.

We take for $n \geqslant 1$

$$\tilde{d}_n(0) = 0. \tag{5.26}$$

Furthermore, let B_n are the Bernoulli polynomials and

$$\delta_n = d_n'(0) - \frac{(-1)^n}{2n} B_{2n}(1/2) \tag{5.27}$$

(the first even Bernoulli polynomials are: $B_0 = 1, B_2(x) = x^2 - x + \frac{1}{6}, B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}, \dots$).

We have $\delta_1 = \delta_2 = 0$ (it is the result of the direct calculation; may be, the same is true for all n).

Let $N \geqslant 3$ be the first integer for which $\delta_N \neq 0$.

If there is such $N, N \geqslant 3$, we take the boundary values for $\tilde{c}_n$ from the equation

$$\tilde{c}_n(0) + \frac{1}{2}\tilde{d}_n'(0) = \frac{\delta_{n+N}}{\delta_N}, n \geqslant 1. \tag{5.28}$$

Lemma 5.3. *Let r be positive and sufficiently large. Then for any fixed integer $M \geqslant 1$ we have uniformly in $\xi \geqslant 0$*

$$\begin{aligned}(2\tanh\xi/2)^{-1/2} A_{11}(r, \sinh\xi/2; 1/2, 1/2) = -\frac{\pi}{2} V_{11,M} + \frac{\delta_N}{r^{2N}} \widetilde{V}_{11,M-N} + \\ + O(\min(\sqrt{\xi}(|\log\frac{1}{r\xi}| + 1), \frac{1}{\sqrt{r}}))r^{-2M-2}\end{aligned} \tag{5.29}$$

where the second term is zero for $M \leqslant N$.

Really, the unknown solution must be the linear combination of two known asymptotic solutions; for some $C_0(r), C_1(r)$ we have

$$(2\tanh\xi/2)^{-1/2} A_{11}(r, \sinh\xi/2; 1/2, 1/2) = C_0(r) W + C_1(r)\widetilde{W}. \qquad ((5.30))$$

If $\xi \to 0$ we have from (4.16) (after taking the limiting value at $\nu = 1/2$)

$$(2\tanh\xi/2)^{-1/2} A_{11}(r, \sinh\xi/2; 1/2, 1/2) = -\sqrt{\xi}\Big(\log\frac{r\xi}{2} - \frac{\Gamma'}{\Gamma}(1) + \\ + \frac{1}{2}\big(\frac{\Gamma'}{\Gamma}(1/2+ir) + \frac{\Gamma'}{\Gamma}(1/2-ir) - 2\log r\big) + O(\xi^2\log\frac{1}{\xi})\Big) \qquad (5.31)$$

Here the sum of logarithmic derivatives of gamma-function is the known asymptotic series

$$\frac{1}{2}\left(\frac{\Gamma'}{\Gamma}(1/2+ir) + \frac{\Gamma'}{\Gamma}(1/2-ir) - 2\log r\right) = -\frac{1}{24r^2} + \ldots + \frac{(-1)^n B_{2n}(1/2)}{2nr^{2n}} + \ldots. \qquad (5.32)$$

On the other side we have for the case $\xi \to 0$ the asymptotic series

$$W = \frac{2}{\pi}\Big((\log\frac{r\xi}{2} - \frac{\Gamma'}{\Gamma}(1))\sum_{n\geqslant 0}\frac{2c_n(0) + d'_n(0)}{2r^{2n}} + \sum_{n\geqslant 1}\frac{d'_n(0)}{r^{2n}} + \\ + O(\xi^2\log\frac{1}{\xi})\Big), \qquad (5.33)$$

$$\widetilde{W} = \sqrt{\xi}\sum_{n\geqslant 0}\frac{2\tilde{c}_n(0) + \tilde{d}'_n(0)}{2r^{2n}} + O(\xi^2) \qquad (5.34)$$

To have the same asymptotic series in front of $(\log\frac{r\xi}{2} - \frac{\Gamma'}{\Gamma}(1))$ we must take $C_0 = -\frac{\pi}{2}$ in (5.30) and define the initial value for $c_n(\xi)$ by the equality (5.19). Then for $C_1(r) = O(r^{-6})$ we have the same coefficients in front of r^{-2} and r^{-4}. If $M = 2$ the process is finished. For $M > 2$ the additional term with $\tilde{V}_{11}$ may be required (I assume that $\delta_n = 0$ for all $n \geqslant 1$ and there is no need to add this term). In this case we take $C_1(r) = \delta_N\, r^{-2N}$ in (5.30) (assuming $\delta_N \neq 0$) and after that we can determine $\tilde{c}_n(0)$ by the equality (5.28).

As before, we omitt the proof of the known fact of nearness of the solutions (5.23) to the corresponding Bessel functions.

5.4. The kernel $A_{10}(r,x;1/2,1/2)$.

For this kernel we will give three different asymptotic formulas (for the intervals $[0,1-\delta),(\delta,1]$ and $[1,\infty)$).

We begin from the differential equations.The function

$$w=(2\tan\xi/2)^{1/2}A_{01}(r,\cos\xi/2;1/2,1/2)$$

satisfies to the equation

$$w''+(-r^2+\frac{1}{4\sin^2\xi})w=0 \tag{5.35}$$

and the function $\tilde{w}=(2\tanh\xi/2)^{1/2}A_{10}(r,\cosh\xi/2;1/2,1/2)$ is the solution of the equation

$$\frac{d^2\tilde{w}}{d\xi^2}+(r^2+\frac{1}{4\sinh^2\xi})\tilde{w}=0. \tag{5.36}$$

For the case $0\leqslant\xi\leqslant\pi-\delta$ with a fixed (small) $\delta>0$ the following simple formula would be sufficient.

Lemma 5.4. *Let $r\to+\infty$; then for any fixed $\delta\in(0,\pi/2)$ we have, uniformly in $0\leqslant\xi\leqslant\pi-\delta$,*

$$(2\tan\xi/2)^{1/2}A_{10}(r,\cos\xi/2;1/2,1/2)=\frac{\pi}{\cosh\pi r}\sqrt{\xi}I_0(r\xi)(1+O(\frac{1}{r})). \tag{5.37}$$

The case $0<\delta\leqslant\xi\leqslant\pi$ is more complicated. As before, we define the sequence $\{a_n,b_n\}$ by the recurrent relations: $a_0\equiv 1, b_0\equiv 0$, and

$$b'_{n+1}=\frac{1}{4\xi}(\frac{b_n}{\xi})'-\frac{1}{2}(a''_n-f_2a_n), n\geqslant 0, \tag{5.38}$$

$$a'_n=-\frac{1}{2}(b''_n-f_2b_n), n\geqslant 1, \tag{5.39}$$

where

$$f_2=\frac{1}{4}(\frac{1}{\xi^2}-\frac{1}{\sinh^2\xi}). \tag{5.40}$$

We take the following initial coditions for these equations:

$$b_n(0)=0, a_n(0)=-\frac{1}{2}b'_n(0)=\frac{(-1)^{n+1}}{4n}B_{2n}(1/2). \tag{5.41}$$

Lemma 5.5. *Let the positive r be sufficiently large; then for any fixed $M \geqslant 1$ we have for $0 \leqslant \xi \leqslant \pi - \delta$ with any fixed $\delta \in (0, \pi/2)$*

$$(2\tan\xi/2)^{-1/2} A_{10}(r, \sin\xi/2; 1/2, 1/2) = \sqrt{\xi} K_0(r\xi) \sum_{0 \leqslant n \leqslant M} \frac{a_n}{r^{2n}} +$$

$$+ (\sqrt{\xi} K_0(r\xi))' \sum_{1 \leqslant n \leqslant M} \frac{b_n}{r^{2n}} + O(\min(\sqrt{\xi}(|\log\frac{1}{r\xi}|) + 1), \frac{1}{\sqrt{r}} e^{-r\xi}) r^{-2M-2} \quad (5.42)$$

Really, the function on the left side must be near to the combination

$$\sqrt{\xi} K_0(r\xi)(1 + O(\frac{1}{r})) + C(r)\sqrt{\xi} I_0(r\xi)(1 + O(\frac{1}{r})) \quad (5.43)$$

since for $\xi \to 0$ we have (it is the limiting case (4.13) as $\nu \to 1/2$)

$$(2\tan\xi/2)^{-1/2} A_{10}(r, \sin\xi/2; 1/2, 1/2) = -\sqrt{\xi} \Bigg(\log\frac{r\xi}{2} - \frac{\Gamma'}{\Gamma}(1) +$$

$$+ \frac{1}{2} \big(\frac{\Gamma'}{\Gamma}(1/2 + ir) + \frac{\Gamma'}{\Gamma}(1/2 - ir) - 2\log r \big) + O(\xi^2 \log\frac{1}{\xi}) \Bigg) \quad (5.44)$$

and

$$K_0(r\xi) = -\log\frac{r\xi}{2} + \frac{\Gamma'}{\Gamma}(1) + O(\xi^2 \log\frac{1}{\xi}). \quad (5.45)$$

Taking in (5.43) $\xi = \pi - \delta$ and $\xi = \delta$ in (5.34) with a fixed small δ we come to the equality

$$O(e^{-(\pi-\delta)r}) = O(e^{-(\pi-\delta)r}) + C(r) e^{(\pi-\delta)r}(1 + O(\frac{1}{r})), \quad (5.46)$$

so for any fixed $\delta > 0$ we have

$$|C(r)| \ll e^{-2(\pi-\delta)r}. \quad (5.47)$$

It means the second term in (5.43) is exponentially small; now the comparison of the asymptotic series at $\xi = 0$ gives us (5.41).

By the way, we come to the new (and very unexpected) method for the calculation of the Bernoulli numbers. Since for $n \geqslant 1$

$$B_{2n}(0) = B_{2n} = \frac{2^{2n-1}}{2^{2n-1} - 1} B_{2n}(1/2) \quad (5.48)$$

we have for $n \geqslant 1$

$$B_{2n} = (-1)^n \frac{2^{2n}}{2^{2n-1}-1}\, n\, b_n'(0), \tag{5.49}$$

where b_n are defined by the recurrent relations (5.38)–(5.39).

Finally, we consider our kernel when argument is larger than 1. Now we have that solution of (5.36) which is

$$\frac{\pi}{\cosh \pi r}\sqrt{\xi}(1+O(\xi^2))$$

when $\xi \to 0$. This boundary condition gives us, uniformly in $\xi \geqslant 0$,

$$(2\tanh \xi/2)^{1/2} A_{10}(r, \cosh \xi/2; 1/2, 1/2) = \frac{\pi}{\cosh \pi r}\Big(\sqrt{\xi} J_0(r\xi) + \\ + \frac{1}{r} O(\min(\sqrt{\xi}, \frac{1}{\sqrt{r}}))\Big). \tag{5.50}$$

5.5. The kernel $A_{00}(u, x; 1/2, \nu)$

First of all, the function

$$v = \sqrt{\sinh \xi}\, A_{00}(u, \frac{1}{\cosh \xi/2}; 1/2, \nu)$$

satisfies to the equation

$$v'' + (t^2 + \frac{1}{4\xi^2})\, v = f_3\, v, \tag{5.51}$$

where

$$f_3 = \frac{1}{4}(\frac{1}{\xi^2} - \frac{1}{\sinh^2 \xi}) + \frac{u^2}{\cosh^2 \xi/2}. \tag{5.52}$$

Furthermore, taking the limiting case $s = 1/2$ in (4.9) we have as $\xi \to 0$:

$$v = -2\sqrt{\xi}\Bigg(\log\frac{t\xi}{2} - \frac{\Gamma'}{\Gamma}(1) + \\ + \frac{1}{4}(\frac{\Gamma'}{\Gamma}(\nu+iu) + \frac{\Gamma'}{\Gamma}(\nu-iu) + \frac{\Gamma'}{\Gamma}(1-\nu+iu) + \frac{\Gamma'}{\Gamma}(1-\nu-iu) - 4\log t) + O(\xi^2 \log\frac{1}{\xi})\Bigg) \\ = -2\sqrt{\xi}\Bigg(\log\frac{t\xi}{2} - \frac{\Gamma'}{\Gamma}(1) + \\ + \sum_{n\geqslant 1}(-1)^{n-1}\frac{B_{2n}(1/2+iu) + B_{2n}(1/2-iu)}{4n\, t^{2n}} + O(\xi^2 \log\frac{1}{\xi})\Bigg) \tag{5.53}$$

(here B_{2n} are the Bernoulli polynomials again).

The equation (5.51) has the same type what we had considered in subsection 5.3. As before, we define the coefficients p_n, q_n by the relations $p_0 \equiv 1, q_0 \equiv 0$ and

$$q'_{n+1} = \frac{1}{2}(p''_n - \frac{1}{2\xi}(\frac{q_n}{\xi})' - f_3\, p_n,\ n \geqslant 0, \tag{5.54}$$

$$p'_n = -\frac{1}{2}(q''_n - f_3\, q_n),\ n \geqslant 1. \tag{5.55}$$

The initial values for these relations we take in the form

$$q_n(0) = 0, p_n(0) = -\frac{1}{2}q'_n(0), n \geqslant 1. \tag{5.56}$$

Now we have

$$q'_1(0) = \frac{1}{2}\, f_3(0) = \frac{1}{4}(B_2(1/2 + iu) + B_2(1/2 - iu)), \tag{5.57}$$

$$q'_2(0) = \frac{1}{6}\, f''_3(0) - \frac{1}{4}\, {f_3}^2(0) = -\frac{1}{8}(B_4(1/2 + iu) + B_4(1/2 - iu)). \tag{5.58}$$

This coincidence (may be, it will be true for $n \geqslant 3$ also) allows us, as in 5.3, write the following asymptotic formula.

Lemma 5.5. *Let $t \to +\infty$ and let u be real and for any fixed $\varepsilon > 0$ $|u| \ll t^\varepsilon$; then, uniformly in $\xi \geqslant 0$, we have*

$$\sqrt{\xi}A_{00}(u, \frac{1}{\cosh \xi/2}; 1/2, \nu) = \frac{1}{\pi}\sqrt{\xi}\, Y_0(t\xi)(1 + \frac{p_1(\xi)}{t^2} + \frac{p_2(\xi)}{t^4})+$$
$$+ \frac{1}{\pi}(\sqrt{\xi}\, Y_0(t\xi))'(\frac{q_1(\xi)}{t^2} + \frac{q_2(\xi)}{t^4}) + O(t^{-6}) \min(\sqrt{\xi}(|\log t\xi| + 1), \frac{1}{\sqrt{t}}) \tag{5.59}$$

Now we consider the same kernel for the case when argument is larger than 1.

The function $v = \sqrt{\sin \xi} A_{00}(u, \frac{1}{\sin \xi/2}; 1/2, \nu)$ (and the same function with ξ replaced by $\pi - \xi$) satisfies to the differential equation

$$v'' + (-t^2 + \frac{u^2}{\sin^2 \xi/2} + \frac{1}{4\sin^2 \xi})v = 0; \tag{5.60}$$

if $\xi \to 0$ we find from (4.8)

$$v = \frac{\Gamma(\nu + iu)\Gamma(1 - \nu + iu)}{2\Gamma(1 + 2iu)}\sqrt{\xi}\, (\xi/2)^{2iu}(1 + O(\xi^2))+$$
$$+ \{\text{the same with } (-u)\} \tag{5.61}$$

It is sufficient to assert that we have

Lemma 5.6. *Let $t \to +\infty$ and u be real with condition $|u| \ll t^{\varepsilon}$ for any fixed $\varepsilon > 0$. Then we have for $0 \leqslant \xi \leqslant \pi - \delta$ with any fixed $\xi \in (0, \frac{\pi}{2})$*

$$\sqrt{\sin}\xi\, A_{00}(u, \frac{1}{\sin\xi/2}; 1/2, \nu) =$$
$$= \mathrm{Re}\{t^{-2iu}\Gamma(\nu + iu)\Gamma(1 - \nu + iu)\sqrt{\xi}\, I_{2iu}(t\xi)(1 + O(\frac{1}{t}))\} \quad (5.62)$$

Finally, in the neighbourhood $\xi = \pi$ we have other asymptotic expansion.

Lemma 5.7. *Let $\{\tilde{p}_n, \tilde{q}_n\}$ are defined by the relations: $\tilde{p}_0 \equiv 1, \tilde{q}_0 \equiv 0$, and*

$$\tilde{q}'_{n+1} = \frac{1}{2}(\frac{1}{2\xi}(\frac{\tilde{q}_n}{\xi})' - \tilde{p}''_n + f_4 p_n),\ n \geqslant 0, \quad (5.63)$$

$$\tilde{p}'_n = -\frac{1}{2}(\tilde{q}''_n - f_4\tilde{q}_n),\ n \geqslant 1, \quad (5.64)$$

where

$$f_4 = \frac{1}{4}(\frac{1}{\xi^2} - \frac{1}{\sin^2\xi}) - \frac{u^2}{\cos^2\xi/2}; \quad (5.65)$$

let the initial values are taken as

$$\tilde{q}_n(0) = 0, \tilde{p}_n(0) = -\frac{1}{2}\tilde{q}'_n(0) = \frac{(-1)^n}{4n}(B_{2n}(1/2 + iu) + B_{2n}(1/2 - iu)). \quad (5.66)$$

Then for any fixed $\delta \in (0, \pi/2)$ we have for $0 \leqslant \xi \leqslant \pi - \delta$ the following asymptotic series

$$\sqrt{\sin}\xi A_{00}(u, \frac{1}{\cos\xi/2}; 1/2, \nu) =$$
$$= 2\sqrt{\xi}\, K_0(t\xi)\sum_{n\geqslant 0}\frac{\tilde{p}_n}{t^{2n}} + 2(\sqrt{\xi}\, K_0(t\xi))'\sum_{n\geqslant 1}\frac{\tilde{q}_n}{t^{2n}}. \quad (5.67)$$

To receive this expansion we use (5.53) and repeat the considerations of subsection 5.4.

5.6. The kernel $A_{10}(u, x; 1/2, \nu)$

Using (4.4) again we see: the function

$$v = (2\tanh\xi/2)^{1/2}(\cosh\xi/2)^{1-2\nu}A_{10}(u, \frac{1}{\cosh\xi/2}; 1/2, \nu) \quad (5.68)$$

is that solution of the differential equation

$$v'' + (t^2 + \frac{1}{4\sinh^2\xi} - \frac{u^2}{\cosh^2\xi/2})v = 0, \tag{5.69}$$

which equals to (we use (4.12) with $s = 1/2, \nu = 1/2 + it, r = u$)

$$2\pi \frac{\cosh \pi t \, \cosh \pi u}{\cosh 2\pi t + \cosh 2\pi u} \sqrt{\xi}\,(1 + O(\xi^2)) \tag{5.70}$$

when $\xi \to 0$.

We have the same equation what we had in 5.5, so we write the full uniform asymptotic expansion with the main term

$$4\pi\, e^{-\pi t} \cosh \pi u \,\sqrt{\xi}\, J_0(t\xi), \quad \xi \geqslant 0. \tag{5.71}$$

For the case when $x > 1$ (but x is not very large) we have the exponentially small function again (with the modified Bessel function instead of J_0).

The function

$$\tilde{v} = (2\tan\xi/2)^{1/2}(\cos\xi/2)^{1-2\nu} A_{10}(u, \frac{1}{\cos\xi/2}; 1/2,\, \nu) \tag{5.72}$$

satisfies to the equation

$$\tilde{v}'' + (-t^2 + \frac{u^2}{\cos^2\xi/2} + \frac{1}{4\sin^2\xi})\tilde{v} = 0. \tag{5.73}$$

Together with initial conditions at $\xi = 0$ it gives us the similar asymptotic formula for $0 \leqslant \xi \leqslant \pi - \delta$

$$(2\tan\xi/2)^{1/2}(\cos\xi/2)^{1-2\nu} A_{10}(u, \frac{1}{\cos\xi/2}; 1/2, \nu) \approx 4\pi\, e^{-\pi t}\, \cosh \pi u \sqrt{\xi}\; I_0(t\xi) \tag{5.74}$$

for any fixed $\delta \in (0, \pi/2)$.

Finally, if x be sufficiently large (it means $0 < \delta \leqslant \xi \leqslant \pi$ in(5.72)) then the solution must be a combination of two modified Bessel's functions.

The unique possibility which have been agreed with (5.74) is the Mcdonald function; so we have for $0 \leqslant \xi \leqslant \pi - \delta$ with any fixed $\delta \in (0, \pi/2)$

$$(2\tan\xi/2)^{1/2}(\sin\xi/2)^{1-2\nu} A_{10}(u, \frac{1}{\sin\xi/2}; 1/2,\, \nu) \approx 4\cosh\pi u\sqrt{\xi}\ K_{2iu}(t\xi). \quad (5.76)$$

5.7. The kernel $A_{11}(u, x; 1/2,\, \nu)$

For this case we have the differential equation

$$w'' + (t^2 + \frac{u^2}{\sinh^2\xi/2} + \frac{1}{4\sinh^2\xi})w = 0$$

for the function

$$w = \sqrt{\sinh\xi} A_{11}(u, \frac{1}{\sinh\xi/2}; 1/2,\, \nu). \quad (5.77)$$

If $\xi \to 0$ we see (it follows from (4.15))

$$w = \frac{\pi e^{-\pi t}}{2\sinh\pi u}\left((\frac{t\xi}{2})^{2iu} \frac{(1 + O(\xi^2))}{\Gamma(1+2iu)} - (\text{the same with } -u)\right). \quad (5.78)$$

So this kernel is exponentially small and we have for the main term of the uniform asymptotic expansion:

$$\sqrt{\sinh}\xi A_{11}(u, \frac{1}{\sinh\xi/2}; 1/2, \nu) \approx \frac{\pi e^{-\pi t}}{2\sinh\pi u}\sqrt{\xi}\left(J_{2iu}(t\xi) - J_{-2iu}(t\xi)\right). \quad (5.79)$$

5.8. The kernel $A_{01}(r, x; 1/2,\, 1/2)$

For the function

$$v = (2\tanh\xi/2)^{-1/2} A_{01}(r, \sinh\xi/2; 1/2,\, 1/2) \quad (5.80)$$

we have the differential equation

$$v'' + (r^2 + \frac{1}{4\sinh\xi^2})v = 0 \quad (5.81)$$

with the initial condition (it follows from (4.11))

$$v = \frac{\pi}{2\cosh\pi r}\sqrt{\xi}(1 + O(\xi^2)), \xi \to 0.$$

It means this function is exponentially small for r large; we have the following main term for the uniform expansion

$$(2\tanh\xi/2)^{-1/2}A_{01}(r,\sinh\xi/2;1/2,\ 1/2)\approx\frac{\pi}{2\cosh\pi r}\sqrt{\xi}\ J_0(r\xi),\xi\geqslant 0. \quad (5.82)$$

5.9. The kernel $A_{00}(r,x;1/2,\ 1/2)$

For this case we have three different expansions.

Firstly we consider the case $0\leqslant\xi<1$. For the function

$$v=(2\tan\xi/2)^{-1/2}A_{00}(r,\sin\xi/2;1/2,\ 1/2) \quad (5.83)$$

we have the differential equation

$$v''+(-r^2+\frac{1}{4\sin^2\xi})v=0 \quad (5.84)$$

and the initial condition (see (4.5) with $s=1/2,\nu=1/2$)

$$v=\frac{\pi}{2\cosh\pi r}\sqrt{\xi}(1+O(\xi^2)),\xi\to 0. \quad (5.85)$$

It means this function is exponentially small for $0\leqslant\xi\leqslant\pi-\delta$ with any fixed $\delta\in(0,\pi/2)$:

$$(2\tan\xi/2)^{-1/2}A_{00}(r,\sin\xi/2;1/2,\ 1/2)\approx\frac{\pi}{2\cosh\pi r}\sqrt{\xi}\ I_0(r\xi)(1+O(\frac{1}{r})). \quad (5.86)$$

The same equation (5.84) we have for the function

$$\tilde{v}=(2\tan\xi/2)^{1/2}A_{00}(r,\cos\xi/2;1/2,\ 1/2); \quad (5.87)$$

if $\xi\to 0$ we have from (4.9) (as the limiting case $s=1/2$)

$$\tilde{v}=2\sqrt{\xi}\Big(-\log\frac{r\xi}{2}+\frac{\Gamma'}{\Gamma}(1)- \\ -\frac{1}{2}(\frac{\Gamma'}{\Gamma}(1/2+ir)+\frac{\Gamma'}{\Gamma}(1/2-ir)-2\log r)\Big)+O(\xi^{5/2}\log\frac{1}{\xi}). \quad (5.88)$$

Furthermore, if $\xi \geqslant \xi_0$ this function must be $O(e^{-\xi_0 r})$. So we have for any $\delta \in (0, \pi/2)$

$$(2\tan\xi/2)^{1/2} A_{00}(r, \cos\xi/2; 1/2,\ 1/2) \approx 2\sqrt{\xi}\ K_0(r\xi),\ \ 0 \leqslant \xi \leqslant \pi - \delta. \tag{5.89}$$

Finally, we have the equation

$$w'' + (r^2 + \frac{1}{4\sinh^2\xi})w = 0 \tag{5.90}$$

for the function

$$w = (2\tanh\xi/2)^{1/2} A_{00}(r, \cosh\xi/2; 1/2,\ 1/2).$$

If $\xi \to 0$ we have the same expansion (5.88) for $w(\xi)$; for this reason

$$(2\tanh\xi/2)^{1/2} A_{00}(r, \cosh\xi/2; 1/2,\ 1/2) \approx -\pi\sqrt{\xi}\ Y_0(r\xi). \tag{5.92}$$

Of course, it is possible to replace the right side by the more detailed asymptotic expansion of the form

$$-\pi\{\sqrt{\xi}\ Y_0(r\xi)\sum_{n\geqslant 0}\frac{a_n}{r^{2n}} + (\sqrt{\xi}\ Y_0(r\xi))'\sum_{n\geqslant 1}\frac{b_n}{r^{2n}}\}, \tag{5.93}$$

where $a_0 \equiv 1, b_0 \equiv 0$ and the recurrent relations for these coefficients may be written from the condition that this function is the formal solution of (5.90).

§6. THE AVERAGING ON THE RIGHT SIDE

In this section we estimate the integrals

$$M_{j,k}(T) = \frac{1}{i}\int\limits_{(1/2)} \mathcal{H}_j(\rho+it)\mathcal{H}_j(\rho-it)\times$$

$$\times\, h_k(\varkappa_j; 1/2, \nu; \rho, 1/2)\omega_T(\rho)d\rho,\quad k = 0, 1. \tag{6.1}$$

The analogous integral for the case of the discret spectrum is

$$\frac{1}{i}\int\limits_{(1/2)} \mathcal{Z}(\rho; \nu, 1/2+ir)(h_0+h_1)(r; 1/2, \nu; \rho, 1/2)\omega_T(\rho)\,d\rho.$$

For some cases the line of integration $\mathrm{Re}\rho = 2$ will be taken instead of the initial line $\mathrm{Re}\rho = 1/2$; on the line $\mathrm{Re}\rho = 2$ the result of integration will be expressed in the explicite form.

The contribution of four poles of the function $\mathcal{Z}(\rho; \nu, 1/2+ir)$ will be considered separatly in subsection 6.7; for all others there is no difference in the consideration of the discret and continuous spectrum.

6.1.The integrals $M_{j,0}; \varkappa_j \geqslant 2T_0$.

It is sufficient consider the case $\varkappa \leqslant 2T_0$, since for large r we have the following inequality.

Proposition 6.1. *Let $r \geqslant 2T_0$; then*

$$|h_0(r; 1/2, \nu; \rho, 1/2)| \ll \frac{1}{r^6}. \tag{6.2}$$

To receive this bound it will be sufficient estimate the absolute value of the integrand in (2.19).

For the case $s = \mu = 1/2$, $\nu = 1/2 + it$, $\rho = 1/2 + i\tau$ (we assume $t = o(\tau^{1/4})$) the integrand in (2.19) is not larger than (we write $w = 1/2 + i\eta$, $\eta \gg 1$)

$$\exp(-\frac{\pi}{2}(\eta + r + |\eta - r| + |\eta - \tau + t| + |\eta - \tau - t|) \\ \max(e^{\pi|\tau - 2\eta|}, e^{\pi\tau})|\hat{\Phi}(1 - 2i\tau + 2i\eta)|. \tag{6.3}$$

Here we can assume that $|\tau - T_0| \ll T \log T_0$ (because of $\omega_T(\rho)$ in (6.1) is exponentially smoll for $|\rho - iT_0| \gg T$).

So for $r \geqslant 2T_0$ the integrand in (2.19) is $O(\exp(-\pi(r - \eta)))$ if $\eta < r$. For this reason we can integrate over the line $\eta \geqslant r - 2\log r$.

But for $\eta \geqslant (1 - \delta)r$, $r \geqslant 2T_0$, for any fixed small $\delta > 0$ we have

$$|\hat{\Phi}(1 - 2i\tau + 2i\eta)| \ll \eta^{-5}, \quad |\gamma(\rho - w, \nu)| \ll \eta^{-1}, \quad |\gamma(1/2 - w, 1/2)| \ll \eta^{-1} \tag{6.4}$$

and we come to the bound (6.2).

It follows from (6.2) that the contribution of the terms with $\varkappa_j \geqslant 2T_0$ to the average on the right side (2.57) is $o(1)$ for the case of our specialization. Really, from the fuctional equation one can see that $|\mathcal{H}_j(\rho \pm it)| \ll \varkappa_j^{1+\varepsilon}$ for the case $\varkappa_j \gg \tau = \operatorname{Im}\rho$. So the result of the averaging of j-th term is $O(T\varkappa_j^{2+2\varepsilon})$. After that we have the sum with terms $O(T\varkappa_j^{-4+2\varepsilon})\alpha_j\mathcal{H}_j^2(1/2)$ and $\varkappa_j \geqslant 2T_0$; this sum is $O(T^{-1+2\varepsilon})$.

6.2.The terms with B_{00} for $r \ll T_0$.

For this case the new representation (4.17) have been used.

Firstly we consider the integral B_{00} and it would be convenient to estimate the function $B_{00}(r, u; \rho, 1/2; 1/2, \nu)$ (we use the symmetry (4.22) for this case).

Lemma 6.1. *Let T_0, T, t are sufficiently large, $T = T_0^{1-\epsilon}$, $t = o(T^{1/4})$; then we have for $r \ll T_0$*

$$\Big| \int\limits_{(1/2)} B_{00}(r,u;\rho,1/2;1/2,\nu)\mathcal{H}_j(\rho+it)\mathcal{H}_j(\rho-it)\omega_T(\rho)\,d\rho\Big| \ll$$

$$\ll \begin{cases} \log^2 T, & r \ll \sqrt{T_0} \\ T\,r^{-2}\log^2 r, & r \gg \sqrt{T_0} \end{cases} \tag{6.5}$$

For the beginning we write in (4.17) (with the variables $(\rho,1/2;1/2,\nu)$)

$$B_{00} = \int_0^{1/\sqrt{2}} + \int_{1/\sqrt{2}}^1 + \int_1^\infty = B^{(1)} + B^{(2)} + B^{(3)},$$

where (after the understandable change of the variable)

$$B^{(1)} = \int_0^{\pi/2} f^{(1)}(\xi;r,u,t)(\sin\xi/2)^{-2\rho}\,d\xi, \tag{6.6}$$

$$B^{(2)} = \frac{1}{2}\int_0^{\pi/2} f^{(2)}(\xi;r,u,t)(\cos\xi/2)^{-2\rho}\,d\xi, \tag{6.7}$$

$$B^{(3)} = \frac{1}{2}\int_0^{\pi/2} f^{(3)}(\xi;r,u,t)(\mathrm{ch}\xi/2)^{-2\rho}\,d\xi. \tag{6.8}$$

In this equalities

$$f^{(1)} = (2\mathrm{tg}\xi/2)^{-1/2}A_{00}(r,\sin\xi/2;1/2,1/2)\sqrt{\sin\xi}A_{00}(u,\frac{1}{\sin\xi/2};1/2,\nu), \tag{6.9}$$

$$f^{(2)} = (2\mathrm{tg}\xi/2)^{1/2}A_{00}(r,\cos\xi/2;1/2,1/2)\sqrt{\sin\xi}A_{00}(u,\frac{1}{\cos\xi/2};1/2,\nu), \tag{6.10}$$

$$f^{(3)} = (2\mathrm{th}\xi/2)^{1/2}A_{00}(r,\mathrm{ch}\xi/2;1/2,1/2)\sqrt{\mathrm{sh}\xi}A_{00}(u,\frac{1}{\mathrm{ch}\xi/2};1/2,\nu); \tag{6.11}$$

it is essential that these functions are not depending in ρ.

Using (5.86), (5.89), (5.62) and (5.67) we see that for any fixed $\delta > 0$ we have

$$B^{(1)} + B^{(2)} = \frac{1}{2} \int_0^{\delta} f^{(2)}(\xi; r, u, t)(\cos \xi/2)^{-2\rho} \, d\xi + O(\exp(-\delta(r+t))) \tag{6.12}$$

(in reality $B^{(1)}$ is $O(e^{-\frac{\pi}{4}(r+t)})$).

The remainder term may be omitted (it gives $o(1)$ into the final result). For the remained integrals we integrate firstly over the variable ρ. Using the Ramanujan integral again we have

$$\begin{aligned}
\frac{1}{i} \int_{(1/2)} & B_{00}(r, u; \rho, 1/2; 1/2, \nu)\mathcal{H}_j(\rho + it)\mathcal{H}_j(\rho - it)\omega_T(\rho) \, d\rho = \\
&= \frac{1}{i} \int_{(2)} (\ldots) \, d\rho = \\
&= T \sum_{n,m \geqslant 1} \frac{t_j(n) t_j(m)}{(nm)^{1/2 + iT_0}} (\frac{n}{m})^{it} \times \\
&\times \Big\{ \int_0^{\delta} f^{(2)}(\xi; r, u, t)\psi(nm \cos^2 \xi/2)(cos\xi/2)^{-1-2iT_0} \, d\xi + \\
&+ \int_0^{\infty} f^{(3)}(\xi; r, u, t)\psi(nm \cosh^2 \xi/2)(\cosh \xi/2)^{-1-2iT_0} \, d\xi \Big\} + O(T_0 e^{-\delta(r+t)}),
\end{aligned} \tag{6.13}$$

where

$$\psi(x) = \left(x^T + x^{-T}\right)^{-1}. \tag{6.14}$$

In (6.13) we have $x = \cos^2 \xi/2$ with small ξ or $x = \cosh^2 \xi/2 \geqslant 1$. For both cases we have the inequality $x \geqslant 2 - \delta$ with small δ if $nm \geqslant 2$. So for $nm \geqslant 2$

$$|\psi(nmx)| \ll a^{-T}, 3/2 < a < 2 - \delta \tag{6.15}$$

and all these terms give $o(1)$ in the final result.

It means we can omit all terms with $nm \geqslant 2$ on the right side (6.13), and take only one term with $n = m = 1$ from this sum.

Now we come to the integrals with the main terms (we use (5.59), (5.67), (5.89) and (5.92))

$$T\int_0^\delta \xi K_0(r\xi)K_0(t\xi)\psi(\cos^2\xi/2)(\cos\xi/2)^{-1-2iT_0}\,d\xi, \tag{6.16}$$

$$T\int_0^\infty \xi Y_0(r\xi)Y_0(t\xi)\psi(\cosh^2\xi/2)(\cosh\xi/2)^{-1-2iT_0}\,d\xi. \tag{6.17}$$

It is sufficient estimate these integrals; the next terms from the corresponding asymptotic expansions for the kernels have the same nature but with additional multipliers r^{-2m} or t^{-2n} with $n, m \geqslant 1$.

In both integrals we can integrate in the interval $0 \leqslant \xi \leqslant \delta(T)$ with $\delta(T) = T^{-1/2}\log T$ only (since the part of our integrals with $\xi \geqslant \delta(T)$ contributes $O(T\exp(-\frac14\log^2 T))$ and this part may be omitted).

We write for $x > 1$

$$\psi(x) = x^{-T}\Big\{\sum_{0\leqslant m\leqslant M-1} x^{-m} + x^{-M}(1+x^{-2T})^{-1}\Big\}$$

and for $x < 1$

$$\psi(x) = x^{T}\Big\{\sum_{0\leqslant m\leqslant M-1} x^{m} + x^{M}(1+x^{2T})^{-1}\Big\}.$$

Let $\mu_m = 1 + 2mT + 2iT_0$; we write the power series

$$(\cos\xi/2)^{\mu_m} = e^{-\mu_m\xi^2/8}\Big(1 + \sum_{k\geqslant 2} a_k(\mu_m)\xi^{2k}\Big). \tag{6.18}$$

Here we can assume $\xi \leqslant (mT)^{-1/2}\log T$ and for this reason we have

$$|a_{2k}(\mu_m)\xi^{4k}| \ll T_0^k(mT)^{-2k}(\log T)^{4k} \ll T_0^{-k(1-2\varepsilon)+\varepsilon}$$

(note that $T = T_0^{1-\varepsilon}$); it means we can omit all terms with $k \geqslant 4$ in (6.18).

Furthermore, we write the power series for $K_0(t\xi)$ in (6.16); since $t\xi \ll tT^{-1/2}\log T \ll T^{-1/4+\varepsilon}$ it is sufficient take the finite number terms (< 4) from this series.

As the result we come to the table integrals

$$T\int_0^\infty K_0(r\xi)e^{-\mu_m\xi^2/8}\xi^{2l+1}\,d\xi,\quad l=0,1,2,\dots \tag{6.19}$$

or

$$T\frac{\partial}{\partial l}\int_0^\infty K_0(r\xi)e^{-\mu_m\xi^2/8}\xi^{2l+1}\,d\xi\Big|_{l=0,1,2,\dots} \tag{6.20}$$

The integral (6.19) equals to

$$\frac{T}{2r}(\frac{8}{\mu_m})^{l+1/2}\Gamma^2(l+1)\exp\left(\frac{r^2}{\mu_m}\right)W_{-l-1/2,0}\left(\frac{2r^2}{\mu_m}\right) \tag{6.21}$$

where $W_{-l-1/2,0}(x)$ denotes the Whittaker function (which exponentially decreseas when $x\to+\infty$).

If $r^2\gg|\mu_m|$ then

$$\Big|\exp(\frac{r^2}{\mu_m})W_{-l-1/2,0}\left(\frac{2r^2}{\mu_m}\right)\Big|\ll\left(\frac{|\mu_m|}{r^2}\right)^{l+1/2} \tag{6.22}$$

so for the main term with $l=0$ we have

$$T\Big|\int_0^\infty K_0(r\xi)e^{\mu_m\xi^2/8}\xi\,d\xi\Big|\ll\frac{T}{r^2},\quad r^2\gg|\mu_m|. \tag{6.23}$$

At the same time for $r^2\ll|\mu_m|$ we have

$$\Big|W_{-l-1/2,0}\left(\frac{2r^2}{\mu_m}\right)\Big|\ll\frac{r}{\sqrt{|\mu_m|}}\log\frac{r}{|\mu_m|} \tag{6.24}$$

and it gives the bound

$$T\Big|\int_0^\infty K_0(r\xi)e^{\mu_m\xi^2/8}\xi\,d\xi\Big|\ll\log r,\quad r^2\ll|\mu_m|; \tag{6.25}$$

the differentiation with respect to r gives the additional multiplier $\log r$.

Exactly by the same way we come to the integrals

$$T\int_0^\infty Y_0(r\xi)\Big\{\log\frac{t\xi}{2}(1+c_1\,(t\xi)^2+\dots)+b_1\,(t\xi)^2+\dots\Big\}e^{-\mu_m\xi^2/8}\xi^{2l+1}\,d\xi \tag{6.26}$$

(we use the power series for $Y_0(t\xi)$ in (6.17)). The leading term is

$$T\int_0^\infty Y_0(r\xi)\exp(-\frac{\mu_m\xi^2}{8})\xi^{2l+1}\,d\xi =$$
$$= \frac{T}{r}(\frac{8}{\mu_m})^{l+1/2}\frac{\exp(-\frac{r^2}{8\mu_m})}{\sin\pi\nu}\big(\Gamma(l+1)\cos\pi l M_{l+1/2,0}(\frac{2r^2}{\mu_m}) - W_{l+1/2}(\frac{2r^2}{\mu_m})\big)$$
$$= \frac{T}{\pi r}(\frac{8}{\mu_m})^{l+1/2}\exp(-\frac{r^2}{8\mu_m})\big((\Gamma'(l+1) - 2\Gamma(l+1)\frac{\Gamma'}{\Gamma}(1))M_{l+1/2,0}(\frac{2r^2}{\mu_m})+$$
$$+ \frac{\partial}{\partial\varepsilon}M_{l+1/2,\varepsilon}(\frac{2r^2}{\mu_m})\Big|_{\varepsilon=0}\big) \qquad (6.27)$$

The last equality follows as the limiting case of the relation between $W_{\varkappa,\varepsilon}$ and $M_{\varkappa,\pm\varepsilon}$: if 2ε is not integer we have

$$W_{\varkappa,\varepsilon}(z) = \frac{\Gamma(-2\varepsilon)}{\Gamma(1/2-\varkappa-\varepsilon)}M_{\varkappa,\varepsilon}(z) + \frac{\Gamma(2\varepsilon)}{\Gamma(1/2-\varkappa+\varepsilon)}M_{\varkappa,-\varepsilon}(z). \qquad (6.28)$$

Now the assertion of Lemma 6.1 follows after the using of the expansion of $M_{l+1/2,0}(z)$ for small and large $|z|$. Since we have the similar asymptotic expansions for the kernels $A_{00}(u,-i(l-1/2);1/2,\nu)$ we come to the same estimates for the integrals wiht $b_{0,l}$.

Lemma 6.2. *Under the same assumptions what we had in the previous lemma we have for* $l\in L$

$$\Big|\int_{(1/2)} b_{0,l}(r;1/2,\nu;\rho,1/2)\mathcal{H}_j(\rho+it)\mathcal{H}_j(\rho-it)\omega_T(\rho)\,d\rho\Big| \ll$$
$$\ll \begin{cases} \log^2 T, & r\ll T_0^{1/2+\varepsilon} \\ T\,r^{-2}\log^2 r, & r\gg T_0^{1/2+\varepsilon}. \end{cases} \qquad (6.29)$$

6.3.The integrals $M_{j,0}$; terms with B_{01}.

The kernel $A_{01}(r,\sinh\xi/2;1/2,1/2)$ is exponentially small for all $\xi\geqslant 0$ (see (5.82)); so all terms with $\varkappa_j\geqslant t^\varepsilon$ with arbitrary small $\varepsilon>0$ give the contribution $o(1)$. For this reason the case $r\ll t^\varepsilon$ have been considered only.

Integrating over the variable ρ in the first line we get again

$$\frac{1}{i}\int\limits_{(1/2)} \mathcal{H}_j(\rho+it)\mathcal{H}_j(\rho-it)B_{01}(r,u;\rho,1/2;1/2,\nu)\omega_T(\rho)\,d\rho =$$

$$T\sum_{N,m\geqslant 1}\frac{t_j(N)\tau_\nu(N)}{(Nm^2)^{1/2+iT_0}}\times$$

$$\times\int_0^\infty A_{01}(r,\sinh\xi/2;1/2,1/2)\times$$

$$\times\sqrt{\sinh\xi}A_{01}(u,\frac{1}{\sinh\xi/2};1/2,\nu)\psi(Nm^2\sinh^2\xi/2)\frac{d\xi}{\sinh\xi/2} \qquad (6.30)$$

(ψ is defined by (6.14)).

If $u=-i(l-1/2), l\geqslant 2$, we have (see (5.14))

$$|\sqrt{\sinh\xi}A_{01}(-i(l-1/2),\frac{1}{\sinh\xi/2};1/2,\nu)| \ll \min(t^3\xi^{7/2},t^{-1/2}). \qquad (6.31)$$

Exactly the same bound we have for the integral

$$\int_{-\infty}^{\infty}\sqrt{\sinh\xi}A_{01}(u,\frac{1}{\sinh\xi/2};1/2,\nu)h(u)\,d\chi(u)$$

because of we can integrate terms with $J_{\pm 2iu}$ from (5.7) over the line $\mathrm{Im}u=\mp 3/2$.

Furthermore (see (5.82)),

$$|(2\tanh\xi/2)^{-1/2}A_{01}(r,\sinh\xi/2;1/2,1/2)| \ll \sqrt{\xi}e^{-\pi r}. \qquad (6.32)$$

These inequalities give the following bound for the integrals in (6.30) with $u=-i(l-1/2)$:

$$\ll e^{-\pi r}\int_0^\infty \min\left((t\xi)^3,(t\xi)^{-1/2}\right)\psi(Nm^2\sinh^2\xi/2)\,d\xi$$

$$\ll e^{-\pi r}(T\sqrt{Nm^2})^{-1}\min\left((\frac{t}{\sqrt{Nm^2}})^3,(\frac{m\sqrt{N}}{t})^{1/2}\right) \qquad (6.33)$$

(we use the fact of the exponential smallness $\psi(x)$ for $|x-1|\geqslant\frac{1}{T}$).

The same estimate we have for the integral with $h(u)$.

As the result we have

Lemma 6.3. *Under assumptions of Lemma 6.1 the integral on the left side (6.30) is* $O(\log^2 t)e^{-\pi r}$ *for* $u=-i(l-1/2), l \geqslant 2$*; the same estimate we have for the result of the integration of this function over* u *with the weight* $h(u)\,d\chi(u)$*.*

Really, for both cases we have the bound

$$\sum_{N,l\geqslant 1} \frac{|t_j(N)\tau_\nu(N)|}{Nl^2}\, e^{-\pi r} \min\Big(\big(\frac{t}{\sqrt{Nl^2}}\big)^3, \big(\frac{l\sqrt{N}}{t}\big)^{1/2}\Big) \tag{6.34}$$

and it rests to note that $|t_j(N)|$ are bounded in average for $\varkappa \ll t^\varepsilon$.

6.4.The integrals B_{10}.

Lemma 6.4. *Under the same conditions we have*

$$\Big|\int_{(1/2)} B_{10}(r,u;\rho,1/2;1/2,\nu)\mathcal{H}_j(\rho+it)\mathcal{H}_j(\rho-it)\omega_T(\rho)\,d\rho\Big| \ll \ e^{-\pi r/2}\log T \tag{6.35}$$

We write B_{10} as $I_1+I_2+I_3$, where

$$I_1 = \int_0^{\pi/2} \sqrt{\sin\xi} A_{00}\left(u, \frac{1}{\sin\xi/2}; 1/2, \nu\right)(2\mathrm{tg}\xi/2)^{-1/2}\times \\ \times A_{10}(r, \sin\xi/2; 1/2, 1/2)(\sin\xi/2)^{-2\rho}\,d\xi, \tag{6.36}$$

$$I_2 = \int_0^{\pi/2} \sqrt{\sin\xi} A_{00}\left(u, \frac{1}{\cos\xi/2}; 1/2, \nu\right)(2\mathrm{tg}\xi/2)^{1/2}\times \\ \times A_{10}(r, \cos\xi/2; 1/2, 1/2)(\cos\xi/2)^{-2\rho}\,d\xi, \tag{6.37}$$

$$I_3 = \frac{1}{2}\int_0^{\infty} \sqrt{\sinh\xi} A_{00}\left(u, \frac{1}{\cosh\xi/2}; 1/2, \nu\right)(2\tanh\xi/2)^{-1/2}\times \\ \times A_{10}(r, \cosh\xi/2; 1/2, 1/2)(\cosh\xi/2)^{-2\rho}\,d\xi. \tag{6.38}$$

Using (5.62) and (5.42) we see that for all r the first integral I_1 is $O(\exp(-\frac{\pi}{2}t))$. At the same time for r's very large we can move the line integration over u to ensure

the estimate $O(r^{-3}\exp(-\frac{\pi}{2}t)$; so the contribution of this term will be exponentally small.

For the second integral we use the asymptotic formulas (5.67) and (5.37). The part of this integral with $\xi \geqslant t^{-1+\varepsilon}$ for any fixed (small) $\varepsilon > 0$ gives the exponentially small contribution again. In the interval $\xi \leqslant t^{-1+\varepsilon}$ (where $\cos\xi/2 = 1 + O(1/t)$) we integrate over ρ under the sign of the integration over ξ. As before, we come to the integral (6.7) and after that we have the integral

$$\frac{T}{\cosh(\pi r)}\int_0^{\delta} \xi K_0(t\xi) I_0(r\xi) e^{-\frac{\pi T}{8}\xi^2}\, d\xi \ll \log T e^{-(\pi-\delta)r}. \tag{6.39}$$

We consider the third integral by the same way; the final integral (we use (5.59) and (5.50)) is

$$\frac{T}{\cosh(\pi r)}\int_0^{\infty} \xi Y_0(t\xi) J_0(r\xi) e^{-\frac{\pi T}{8}\xi^2}\, d\xi \ll\ e^{-\pi r}\log T. \tag{6.40}$$

6.5.Terms with $B_{11}(r,u;\rho,1/2;1/2,\nu)$ $(= B_{10}(r,u;1/2,\nu;\rho,1/2))$.

The estimation of this last integral is more complicated; it would be realized by the following way.

Firstly we consider $B_{11}(r,u;\rho,1/2;1/2,\nu)$ for $r \leqslant T_0 - 2T^{1+\varepsilon}$; for this case B_{11} is exponentially small.

For $r \gg T_0$ the integral B_{10} instead of B_{11} will be considered; as before we integrate over ρ in the first line.

In contrast with the previous integrals the result of this integration is the "long" sum. But we have the opportunity use "the convolution formulas" (that is the replacement of the summation over $\varkappa_j$ by the integration over this variable). Asymptotically the last integrand is the product of two δ-functions with the different supports; it allows us to finish our estimates for the sums over discret and continuous sprectrum.

6.5.1.The exponential case.

Lemma 6.5. *Let T_0, τ, t are sufficiently large, $T = T_0^{1-\varepsilon}$, $t = o(T^{1/4})$; then we have for $r \leqslant T_0 - 2T^{1+\varepsilon}$*

$$\left| \int\limits_{(1/2)} B_{11}(r,u;1/2,\nu;\rho,1/2)\mathcal{H}_j(\rho+it)\mathcal{H}_j(\rho-it)\omega_T(\rho)\, d\rho \right| \ll \exp(-T^{\varepsilon}). \tag{6.41}$$

First of all, we can assume here $|\tau - T_0| \leqslant T^{1+\varepsilon}$ $(= T_0^{1-\varepsilon^2})$, since $|\omega_T(\rho)| \ll \exp\left(-\frac{|\tau-T_0|}{T}\right)$.

Now we use (4.20) (with $s = \mu = 1/2,\ \rho = 1/2 + i\tau,\ \nu = 1/2 + it$).

In this expression (see (4.13) and (4.14)) for $0 \leqslant x < 1$

$$\begin{aligned}(2\pi)^{2\rho-1}A_{10}(r,\sqrt{x};\rho,\nu) = \\ = \frac{\cosh(\pi r)}{2\cos(\pi\nu)}\Bigg\{\frac{\Gamma(\rho+it+ir)\Gamma(\rho+it-ir)}{\Gamma(2\nu)}\times \\ \times x^{\nu}F(\rho+it+ir,\rho+it-ir;2\nu;x)- \\ -\frac{\Gamma(\rho-it+ir)\Gamma(\rho-it-ir)}{\Gamma(2-2\nu)}\times \\ \times x^{1-\nu}F(\rho--it+ir,\rho-it-ir;2-2\nu;x)\Bigg\}\end{aligned} \tag{6.42}$$

and for $x > 1$

$$\begin{aligned}(2\pi)^{2\rho-1}A_{10}(r,\sqrt{x};\rho,\nu) = \\ = \frac{ix^{1/2-\rho}\sin(\pi\nu)}{\sinh(\pi r)}\Bigg\{\frac{\Gamma(\rho+it+ir)\Gamma(\rho-it-ir)}{\Gamma(1+2ir)}\times \\ \times x^{-ir}F(\rho+it+ir,\rho-it+ir;1+2ir;\frac{1}{x})- \\ -\frac{\Gamma(\rho+it-ir)\Gamma(\rho-it-ir)}{\Gamma(1-2ir)}\times \\ \times x^{ir}F(\rho+it-ir,\rho-it-ir;1-2ir;\frac{1}{x})\Bigg\}.\end{aligned} \tag{6.43}$$

After the multiplication by $\omega_T(\rho)$ we have in these expressions the multipliers in front of the hypergeometric functions which are not larger than

$$O(\frac{1}{\sqrt{t}})\exp(-\pi(\tau - t - r + \frac{|\tau - T_0|}{T})). \tag{6.44}$$

If $r \leqslant T_0 - 2T^{1+\varepsilon}$ (note that $T^{1+\varepsilon} = T_0^{1-\varepsilon^2} = o(T_0)$) then this bound is not larger than $O(\exp(-\pi T^{\varepsilon}))$; so it is sufficient to see that these hypergeometric functions are bounded.

For the function

$$F = (1-x)^{\rho} x^{\nu} F(\rho+\nu-1/2+ir, \rho+\nu-1/2-ir; 2\nu; x) \tag{6.45}$$

we have the differential equation

$$F'' + \left(\frac{p(x)}{x^2(1-x)^2} + \frac{1-x+x^2}{4x^2(1-x)^2} \right) F = 0 \tag{6.46}$$

with

$$p(x) = t^2(1-x) + (\tau^2 - r^2(1-x))x; \tag{6.47}$$

this p is positive for all $x \in [0,1]$ if $r \leqslant \tau$. For this reason we have for F the Liouville–Green approximation

$$F \cong \sqrt{t}\frac{x^{1/2}(1-x)^{1/2}}{p^{1/4}} \exp(i\xi), \quad \xi = \int_0^x \left(\frac{\sqrt{p(y)}}{1-y} - t \right) \frac{dy}{y} + t \log x. \tag{6.48}$$

As the consequence we have (we use the Stirling expansion) for $r < \tau - t$

$$|A_{10}(r, \sqrt{x}; \rho, \nu)| \ll e^{-\pi(\tau-t-r)} x^{1/2}, \quad x < 1. \tag{6.49}$$

By the similar way we find for $x > 1$

$$|A_{10}(r, \sqrt{x}; \rho, \nu)| \ll e^{-\pi(\tau-t-r)}. \tag{6.50}$$

Using again the corresponding differential equation we see from (4.10) and (4.11) that

$$\left| A_{01}\left(u, \frac{1}{\sqrt{x}}; 1-\rho, 1/2\right) \right| \ll \begin{cases} 1, & x \ll 1 \\ \frac{1}{\sqrt{x}} \log x, & x \gg 1. \end{cases} \tag{6.51}$$

Now (6.41) follows.

6.5.2.The integration over ρ.

Proposition 6.2.

$$\frac{1}{i}\int\limits_{(1/2)} B_{10}(r,u;1/2,\nu;\rho,1/2)\mathcal{H}_j(\rho+it)\mathcal{H}_j(\rho-it)\omega_T(\rho)\,d\rho =$$

$$= \frac{T}{2\pi}\sum_{N,l\geqslant 1}\frac{t_j(N)\tau_\nu(N)}{(Nl^2)^{1/2+iT_0}}f_{N,l}(r,u) \quad (6.52)$$

where with ψ from (6.14)

$$f_{N,l}(r,u) = \int\limits_0^\infty A_{11}(r,\sqrt{x};1/2,1/2)A_{01}(u,\frac{1}{\sqrt{x}};1/2,\nu)\psi(Nl^2x)x^{-1/2-iT_0}\,dx. \quad (6.53)$$

On the line $\mathrm{Re}\rho = 1/2$ we can change the order of integration. After that we have the integral

$$\int\limits_{(1/2)} \mathcal{H}_j(\rho+it)\mathcal{H}_j(\rho-it)x^{-\rho}\omega_T(\rho)\,d\rho$$

with $x > 0$. Here we can integrate over the line $\mathrm{Re}\rho = 2$, where the Hecke series are absolutely convergent. Using the Ramanujan integral again we come to the expression

$$\sum_{n,m\geqslant 1}\frac{t_j(n)t_j(m)}{(nm)^{1/2+iT_0}}x^{-1/2-iT_0}\psi(nmx)\left(\frac{n}{m}\right)^{it}. \quad (6.54)$$

Now we have

$$t_j(n)t_j(m) = \sum_{l\backslash(n,m)} t_j\left(\frac{nm}{l^2}\right) \quad (6.55)$$

and after the corresponding change n by nl, m by ml and mn by N we get (6.48).

6.5.3. The large N's in (6.52).

First of all we replace the series in (6.52) by the finite sum with $N \leqslant N_0(T)$.

Proposition 6.3. *Let*

$$h_{N,l}(r) = \int\limits_{-\infty}^{\infty} f_{N,l}(r,u)h(u)\,d\chi(u); \quad (6.56)$$

then for $N_0 = t^3\sqrt{T}$ *we have*

$$\sum_{j\geqslant 1} \alpha_j \mathcal{H}_j^2(1/2) \sum_{Nl^2\geqslant N_0} \frac{t_j(N)\tau_\nu(N)}{(Nl^2)^{1/2+iT_0}} h_{N,l}(\varkappa_j) = O(T^\varepsilon) \tag{6.57}$$

and the same estimate is valid for the corresponding integral over the continuous spectrum.

To estimate $|h_{N,l}|$ we rewrite the definition $f_{N,l}$. We do the change of the variable $x \mapsto \frac{1}{Nl^2}\exp(\frac{x}{T})$ in (6.52).

Furthermore, we write instead of the kernel $A_{01}(u,...)$ the function

$$\frac{i\sin\pi\nu}{\sinh\pi u}\frac{\Gamma(\nu+iu)\Gamma(1-\nu+iu)}{\Gamma(1+2iu)} x^{iu} F(\nu+iu, 1-\nu+iu; 1+2iu; -x);$$

(this replacement is possible since $h(u)$ is the even function) and we integrate this function on the line $\operatorname{Im} u = -3/2$. Now we have the representation

$$\begin{aligned} T(Nl^2)^{1/2-iT_0}\, h_{N,l}(r) = \int\limits_{\operatorname{Im} u=-3/2} & \frac{i\sin\pi\nu}{\sinh\pi u}\frac{\Gamma(\nu+iu)\Gamma(1-\nu)}{(Nl^2)^{iu}\Gamma(1+2iu)} h(u)\times \\ & \times \int_{-\infty}^{\infty} A_{11}(r, \frac{1}{l\sqrt{N}}\exp(\frac{x}{2T}); 1/2, 1/2)\times \\ & \times F(\nu+iu, 1-\nu+iu; 1+2iu; -\frac{1}{Nl^2}\exp(-\frac{x}{2T}))\times \\ & \exp(\frac{1+2iu-2iT_0}{2T}x)\frac{dx}{\cosh x}\, d\chi(u) \end{aligned} \tag{6.58}$$

The part of this integral with $|u| \geqslant T^\varepsilon$ or $|x| \geqslant T^\varepsilon$ gives $o(1)$ in the final result.

Since $Nl^2 > t^3\sqrt{T}$ we can use the power series for the hypergeometric functions in (6.58).

For the kernel A_{11} we use the asymptotic expansion (5.29). Let $|x| \leqslant T^\varepsilon$ and N be large; we define the positive ξ by the equation

$$\sinh\xi/2 = \frac{1}{l\sqrt{N}}\exp(\frac{x}{2T}) = \frac{1}{l\sqrt{N}}(1+\frac{x}{2T}+\frac{x^2}{8T^2}+...) \tag{6.59}$$

From (5.29) we find $A_{11} = O(\xi|\log(r\xi)|)$ for $r \leqslant l\sqrt{N}$, $A_{11} = O(\sqrt{\frac{\xi}{r}})$ for $Nl^2 \leqslant r \leqslant T^{1+\varepsilon}Nl^2$ and for $r \geqslant T^{1+\varepsilon}Nl^2$ we have the asymptotic series with the main term

$$A_{11}(r, \sinh\xi/2; 1/2, 1/2) \approx -\frac{1}{\sqrt{\pi}}(\tanh\xi/2)^{1/2}\sinh(r\xi - \frac{\pi}{4}) \tag{6.60}$$

If $r \gg T_0^{1+\varepsilon}l\sqrt{N}$ we can integrate by parts any times. Each integration gives the additional multiplier $(r^{-1})Tl\sqrt{N})$; as the result we have for any $M \geqslant 2$

$$\begin{aligned}\frac{1}{l\sqrt{N}}|h_{N,l}| &\ll \frac{t^3}{TN^3l^6}\log(rNl^2),\ r \ll l\sqrt{N},\\ &\ll \frac{t^3}{T\sqrt{r}(Nl^2)^{11/4}},\quad l\sqrt{N} \ll r \ll T_0^{1+\varepsilon}l\sqrt{N},\\ &\ll \left(\frac{Tl\sqrt{N}}{r}\right)^M \frac{t^3}{T\sqrt{r}(Nl^2)^{11/4}},\ r \gg T_0^{1+\varepsilon}.\end{aligned} \tag{6.61}$$

We have the known estimates ([13], [2]):

$$\sum_{\varkappa_j \leqslant P} \alpha_j \mathcal{H}_j^4(1/2) \ll P^{2+\varepsilon}, \tag{6.62}$$

$$\sum_{\varkappa_j \leqslant P} \alpha_j t_j^2(N) \ll P^2 + N^{1/2+\varepsilon}; \tag{6.63}$$

for this reason we have for $P \gg N^{1/4}$

$$\sum_{\varkappa_j \leqslant P} \alpha_j |\mathcal{H}_j^2(1/2)t_j(N)| \frac{1}{\sqrt{\varkappa_j}} \ll \sqrt{P^{1+\varepsilon}(P^2 + N^{1/2+\varepsilon})} \ll P^{3/2+\varepsilon}. \tag{6.64}$$

It means the sum in (6.57) is not larger than $O(\frac{t^3T^{1/2+\varepsilon}}{N_0})$ for any $\varepsilon > 0$; taking $N_0 = t^3T^{1/2}$ we come to the assertion of Proposition 6.3.

6.5.4. The convolution formula.

Let for a given function f and for an integer $N \geqslant 1$

$$Z_N(f) = \sum_{j\geqslant 1} \alpha_j t_j(N)\mathcal{H}_j^2(1/2) f(\varkappa_j) + \frac{1}{\pi}\int_{-\infty}^{\infty} \tau_{1/2+ir}(N)\frac{|\zeta(1/2+ir)^4}{|\zeta(1+2ir)|^2} f(r)\,dr \tag{6.65}$$

This quadratic form in the Hecke series had been expressed in terms of the convolution of the Fourier coefficients of the Eisenstein series (Theorem 3.1 in [8]; here we take the special case $s = \nu = 1/2$).

Theorem 6.1. *Let $f(r)$ be the even regular function in the strip $|\mathrm{Im}\, r| \leqslant 5/2, f(\pm i/2) = 0$ and $|f(r)| \ll |r|^{-B}$ for some $B > 4$ when $r \to \infty$ in this strip. Then for any integer $N \geqslant 1$ we have*

$$Z_N(f) = \frac{1}{\pi}\frac{d(N)}{\sqrt{N}}\int_{-\infty}^{\infty}(-\log\frac{N}{2\pi} - 2\frac{\Gamma'}{\Gamma}(1/2+ir) - 2\frac{\Gamma'}{\Gamma}(1) + \zeta'(0)) f(r)\,dr -$$
$$- \frac{1}{4\pi}\int_{-\infty}^{\infty} f(r)\frac{d\chi(r)}{\cosh \pi r} + \frac{1}{\sqrt{N}}\sum_{n\neq N} d(n)d(n-N)w_0(\sqrt{\frac{n}{N}}) +$$
$$+ \frac{1}{\sqrt{N}}\sum_{n\geqslant 1} d(n)d(n+N)w_1(\sqrt{\frac{n}{N}}), \tag{6.66}$$

where $d(n)$ is the number of the positive divisors of $|n|$ and two functions w_j are defined by the equalities

$$w_j(x) = \frac{1}{\pi x}\int_{-\infty}^{\infty} A_{0j}(r,x;1/2,1/2) f(r)\,d\chi(r),\ \ j = 0,1 \tag{6.67}$$

6.5.4.The construction of the suitable taste function.

For the case $\varkappa_j \geqslant T_0 - 2T^{1+\varepsilon}$ we use the representation (6.50).

Let us define

$$\Omega(r) = \frac{2\beta}{\pi}\int_{T_0/2}^{\infty}\left((\cosh \beta(r-u))^{-1} + (\cosh \beta(r+u))^{-1}\right) du,\ \ \beta = T_0^{-1+\varepsilon}. \tag{6.68}$$

This even function is regular in the wide strip $|\mathrm{Im}| r < \frac{\pi}{2\beta}$, for real r this function is $O(\exp(-T_0^{\varepsilon}))$ for $|r| \leqslant T_0/4$ and for the positive r with $r \geqslant 3T_0/4$ we have

$$1 - c_1\exp(-c_2 T_0^{\varepsilon}) \leqslant \Omega(r) \leqslant 1 \tag{6.69}$$

with the fixed positive constants c_1, c_2.

Furthermore, let

$$q(r) = \frac{(r^2+1/4)^2(r^2+9/4)^2}{(r^2+1/4)^2(r^2+9/4)^2+M^2}, \tag{6.70}$$

where the fixed positive M be so large that q has no poles in the strip $|\mathrm{Im} r| \leqslant 5/2$.

With these notations we define

$$F_{N,l}(r,u) = q(r)\Omega(r)\, f_{N,l}(r,u) \tag{6.71}$$

($f_{N,l}(r,u)$ is defined by (6.51)).

It is obvious this function satisfies to all conditions of Theorem 6.1.

The main characteristic of this function is the following fact: the result of the summation over the discret and continuous spectrum of the quantities (6.50) we can replace by the sum

$$\frac{T}{2\pi}\sum_{Nl^2\leqslant N_0}\frac{\tau_\nu(N)}{(Nl^2)^{1/2+iT_0}}\, Z_N(F_{N,l}),\ \ N_0 = t^3\sqrt{T}. \tag{6.72}$$

Really, the difference

$$F_{N,l}(r,u) - f_{N,l}(r,u) = (q(r)\Omega(r)-1)f_{N,l}(r,u)$$

after the summation over N, l gives the exponentially small contribution for $r \leqslant T_0/4$ (because of Ω is very small for these values of r and we have case which had been considered in 6.5.1.). But for $r \gg T_0$ this difference is $O(r^{-8})|f_{N,l}(r,u)|$. In any case we have for the integral (6.58) the rough estimate $h_{N,l}(r) = O((rl\sqrt{N})^{-1})$; with the additional multiplier r^{-8} it is sufficient to assert the smallness of the full sum with this difference.

6.5.6. The first sum with w_0 (the case $n \geqslant N+1$).

We use identity (6.66) to estimate $Z_{N,l}(F_{N,l})$; first of all we consider the integral

$$\frac{1}{\sqrt{N}}w_0(\sqrt{\frac{n}{N}}) = \frac{1}{\pi\sqrt{n}}\int_{-\infty}^{\infty} A_{00}(r,\sqrt{\frac{n}{N}};1/2,1/2)\, F_{N,l}(r,u)\, d\chi(r). \tag{6.73}$$

After the replacement $f_{N,l}$ by the integral representation (6.51) (where we do the change of the variable $x \mapsto \sinh^2 \eta/2$) we come to the double integral (we write $\cosh^2 \xi/2$ instead of $\frac{n}{N}$, $\xi > 0$)

$$\frac{1}{\pi\sqrt{n}} \int_{-\infty}^{\infty} A_{00}(r, \cosh \xi/2; 1/2, 1/2) \int_0^{\infty} A_{11}(r, \sinh \xi/2; 1/2, 1/2) \times$$
$$\times A_{01}(u, \frac{1}{\sinh \eta/2}; 1/2, \nu) \psi(Nl^2 \sinh^2 \eta/2)(\sinh \eta/2)^{-2iT_0} \cosh \eta/2 \, d\eta \times$$
$$\times q(r)\Omega(r)\, d\chi(r). \quad (6.74)$$

Let $\delta = T^{-1+\varepsilon}$ with small $\varepsilon > 0$; we write the integral on the right side (6.74) as $v_1 + v_2 + v_3$, where

$$v_1(n,l) = \int_{-\infty}^{\infty} \int_0^{\xi-\delta} (\ldots)\, d\eta\, d\chi(r), v_2 = \int_{-\infty}^{\infty} \int_{\xi-\delta}^{\xi+\delta} (\ldots)\, d\eta\, d\chi(r), \quad (6.75)$$

and in v_3 the integration is doing on $\eta \geqslant \xi + \delta$.

Proposition 6.4. *For $j = 1$ and $j = 3$ we have*

$$\sum_{n \geqslant N+1} d(n)d(n-N)|v_j(n,l)| \ll \exp(-T^{\varepsilon}) \quad (6.76)$$

Firstly we note that for all $n \geqslant N+1$

$$\xi(n) \gg T^{-1/4} t^{-3/2} \quad (6.77)$$

(if n be near to N we have $\cosh \xi/2 = 1 + \frac{1}{8}\xi^2 + \ldots = 1 + \frac{n-N}{N} + \ldots$; so $\xi(n) \gg \frac{1}{\sqrt{N}} \gg N_0^{-1/2}$).

Now for $\eta \geqslant \xi + \delta$ we have $Nl^2 \sinh^2 \eta/2 \geqslant l^2(n - N + \delta\sqrt{N}(n-N))$.

For this reason for $\eta \geqslant \xi + \delta$

$$\psi(Nl^2 \sinh^2 \eta/2) \ll \exp(-l^2(n - N + \delta\sqrt{N}(n-N))T), \quad (6.78)$$

so the series over n is convergent and this sum is $O(\exp(-T^{\varepsilon}))$.

For the case $j = 1$ we have the same result if $\xi(n) = O(1)$.

For large n we have $\xi(n) \geqslant \log \frac{n}{N}$; writing $Y_0(r\xi)$ as $\frac{1}{2i}(H_0^{(1)} - H_0^{(2)})$ we integrate the Hankel function $H_0^{(1)}$ on the line $\operatorname{Im} r = \Delta > 1/2$ and the other function on the line $\operatorname{Im} r = -\Delta < -1/2$ (we assume Δ be near to $1/2$). It gives the convergence of our series and the smallness $\psi(Nl^2 \sinh^2 \eta/2)$ for $\eta \leqslant \xi - \delta$ gives the same result.

It rests estimate the sum with v_2.

Proposition 6.5.

$$\sum_{l^2(n-N)\geqslant 2} d(n)d(n-N)v_2(n,l) \ll \exp(-T^{\varepsilon}) \tag{6.79}$$

Really, in this integral we have $|\xi-\eta| \leqslant T^{-1+\varepsilon}$, so $Nl^2\sinh^2\eta/2$ is near to $l^2(n-N)$.

If $l^2(n-N)\geqslant 2$ we have $\psi(Nl^2\sinh^2\eta/2) \ll \exp(-Tl^2(n-N))$ and any rough estimate for the result of integration over r and η gives the desired result.

Now we consider the integral with $n=N+1$ and $l=1$.

Proposition 6.6. *Let $B(r;\xi,\eta)$ denotes the integrand in (6.74); then for $n = N+1$ and $l=1$ we have*

$$\lim_{P\to\infty}\int_0^P\int_{\xi-\delta}^{\xi+\delta} B(r;\xi,\eta)\,d\eta\,d\chi(r) = \frac{1}{\pi}A_{01}(u,\sqrt{N};1/2,\nu)N^{iT_0}(1+\frac{1}{N})^{1/2}+O(\frac{N}{T^2}) \tag{6.80}$$

We change the order of the integration and integrate over r in the first line. The part of this integral with $r\leqslant T$ gives the exponentially small contribution and may be omitted.

In the interval $T\leqslant r\leqslant P$ the quantity $r\xi$ is large since $\xi \gg T^{-1/4}t^{-3/2}$. For this reason both expansions (5.93) and (5.29) may be rewritten in the trigonometric form. Using the asymptotic expansions for the Bessel functions we have

$$A_{00}(r,\cosh\xi/2;1/2,1/2) = -(\pi r\tanh\xi/2)^{-1/2}\big((1+\mathcal{E}_1)\sin(r\xi-\frac{\pi}{4})+ \\ +\frac{\mathcal{E}_2}{r\xi}\cos(r\xi-\frac{\pi}{4})\big), \tag{6.81}$$

$$A_{11}(r,\sinh\eta/2;1/2,1/2) = \\ = -(\frac{1}{\pi r})^{1/2}\big((1+\mathcal{D}_1)\sin(r\eta)-\frac{\pi}{4})+ \\ +\frac{\mathcal{D}_2}{r\eta}\cos(r\eta-\frac{\pi}{4})\big) \tag{6.82}$$

where $\mathcal{E}_j$ and $\mathcal{D}_j$ are the asymptotic series in the inverse degrees of $r^2,(r\xi)^2$ and $r^2,(r\eta)^2$ correspondingly.

In these expansions we have $\mathcal{E}_1 = O((r\xi)^{-2})$ and $\mathcal{E}_2 = O(1)$ and the similar order have $\mathcal{D}_j$.

As the result we have

$$A_{00}(r, \cosh\xi/2; 1/2, 1/2) A_{11}(r, \sinh\eta/2; 1/2, 1/2) =$$
$$= \frac{1}{2\pi r}\left(\frac{\tanh\eta/2}{\tanh\xi/2}\right)^{1/2} \left\{ \cos r(\xi-\eta)\ (1+\mathcal{E}_3) + \right.$$
$$\left. + \frac{\sinh r(\xi-\eta)}{r}\left((1+\mathcal{E}_1)\frac{\mathcal{D}_2}{\eta} - (1+\mathcal{D}_1)\frac{\mathcal{E}_2}{\xi} + \ldots \right\}, \quad (6.83)$$

where $\mathcal{E}_3$ is the asymptotic series in the inverse degrees of $r^2, (r\xi)^2, (r\eta)^2$ ($\mathcal{E}_3 = O((r\xi)^{-2}) + (r\eta)^{-2}$) and the unwritten terms contain $\cos r(\xi+\eta)$ or $\sin r(\xi+\eta)$.

These unwritten terms contribute $O(T^{-M})$ for any fixed $M \geqslant 4$ (since we can integrate these terms by parts any times).

For large positive r we have $d\chi(r) = \frac{2}{\pi}\, r\,(1+O(e^{-\pi r}))\, dr$ and

$$\int_T^P A_{00}(r, \cosh\xi/2; 1/2, 1/2) A_{11}(r, \sinh\eta/2; 1/2, 1/2) q(r)\Omega(r)\, d\chi(r) =$$
$$= \frac{1}{\pi^2}\left(\frac{\tanh\eta/2}{\tanh\xi/2}\right)^{1/2} \left\{ \frac{\sin P(\xi-\eta)}{\xi-\eta}(1+\mathcal{E}_3(P)) - \right.$$
$$- \int_T^P \frac{\sin r(\xi-\eta)}{\xi-\eta}((1+\mathcal{E}_3)q\Omega\tanh\pi r)'\, dr +$$
$$+ \int_T^P \frac{\sin r(\xi-\eta)}{\xi-\eta}\left(\frac{(1+\mathcal{E}_1)\mathcal{D}_2}{\eta} - \frac{(1+\mathcal{D}_1)\mathcal{E}_2}{\xi}\right) q\Omega\tanh\pi r\, dr +$$
$$\left. + O(\exp(-T)) \right\}. \quad (6.84)$$

After multiplication by $A_{01}(u, \ldots)$ the first term gives in the limit $P \to \infty$

$$A_{01}(u, \frac{1}{\sinh\xi/2}; 1/2, \nu)\cosh\xi/2 (\sinh\xi/2)^{-2iT_0}\psi(N\sinh^2\xi/2) =$$
$$= A_{01}(u, \sqrt{N}; 1/2, \nu) N^{iT_0}(1+\frac{1}{N})^{1/2} \quad (6.85)$$

Note that all asymptotic series $\mathcal{E}_1, \mathcal{E}_2, ...$ allow the term by term differentiation (because of it is sufficient take the finite number terms in asymptotic expansions for our kernels and for expansions of the Bessel function this fact is known very well). In particular, $\mathcal{E}_3'$ is the asymptotic series with the main term $-\frac{9}{64}\frac{1}{r^3\xi^2}$ and $\mathcal{E}_1', \mathcal{D}_2'$ are the asymptotic series with the main terms of the order $r^{-3}\xi^{-2}, r^{-3}\eta^{-2}$ correspondingly. The derivatives of q and Ω have the smaller order.

Using the fact that $\int_P^r u^{-1}\sin au\, du$ is bounded for all positive P, r and a, we find the estimate $O(T^{-2}\xi^{-3})$ for the last integral in (6.85). So the result of the integration over η gives not larger than

$$\max_{|\xi-\eta|\leqslant T^{-1+\varepsilon}} |A_{01}(u, \frac{1}{\sinh\eta/2}; 1/2, \nu)| \times \frac{1}{T^2\xi^3}\int_{\xi-\delta}^{\xi+\delta}\psi(N\sinh^2\eta/2)\, d\eta \ll$$

$$\ll \frac{1}{T^3\xi^3\sqrt{N}}.$$

Finally, the second integral is $O(\frac{1}{T^2\xi^2}) = O(\frac{N}{T^2})$; it gives us (6.80).

6.5.7. The second sum with w_0 ($n \leqslant N-1$).

Proposition 6.7. *Let w_0 be defined by (6.73); then*

$$\sum_{n\leqslant N-1, l\geqslant 1} d(n)d(N-n)w_0(\sqrt{\frac{n}{N}}) \ll \exp(-T^{3/4}t^{-3/2}) \tag{6.86}$$

Really, we have (see (5.89))

$$A_{00}(r, \cos\xi/2; 1/2, 1/2) \approx \sqrt{\frac{2\xi}{\tanh\xi/2}}\, K_0(r\xi),\ 0 \leqslant \xi \leqslant 3\pi/4. \tag{6.87}$$

Let ξ be defined by the equation $\cos\xi/2 = \sqrt{\frac{n}{N}}$; then we have $\xi \gg \sqrt{\frac{N-n}{N}} \gg N^{-1/2}$ and it means that $\xi \gg T^{-1/4}t^{-3/2}$ since $N \leqslant N_0 = T^{1/2}t^3$.

We can assume $r \gg T$; then $r\xi \gg T^{1/4-\varepsilon}t^{-3/2}$ and (6.86) follows after any rough estimate of the result of summation and integration.

6.5.8. The sum with w_1.

Proposition 6.8. *Let w_1 be defined by (6.67) with $F_{N,l}$ instead of f; then*

$$\sum_{n\geqslant 1} d(n)d(n+N)w_1(\sqrt{\frac{n}{N}}) \ll \exp(-T). \tag{6.88}$$

This estimate follows from (5.82) and the definition of $\Omega(r)$. We have $\Omega(r) \ll \exp(-T)$ for $r \leqslant T$ and at the same time

$$A_{01}(r, \sinh\xi/2; 1/2, 1/2) \approx \frac{\pi}{2\cosh \pi r}\sqrt{\xi}J_0(r\xi) \ll \exp(-\pi r). \tag{6.89}$$

It is sufficient for $n \leqslant NT^M$ for any fixed $M \geqslant 2$. For very large n we integrate the first term from (4.10) on the line Im $= -1/2 - \varepsilon_0$ and the second one on the line $\operatorname{Im} r = 1/2 + \varepsilon_0$ with small positive ε_0. Due to the multipliers $(\frac{n}{N})^{\pm ir}$ in the denominator it ensures the convergence and we come to (6.86) again.

6.5.9. The first term of the convolution.

Proposition 6.9. *Let $W_{N,l}$ denotes the first integral on the wright side (6.66) for the case $f = F_{N,l}(r)$; then we have*

$$|W_{N,l}| \ll \frac{\log T}{T^3}. \tag{6.90}$$

The part of this integral with $|r| \leqslant T$ is exponentially small because of the smallness Ω. For $r \gg T$ we use the asymptotic expansion (5.29),

$$A_{11}(r, \sinh\xi/2; 1/2, 1/2) = -\frac{\pi}{2}\sqrt{2\tanh\xi/2}\big\{\sqrt{\xi}Y_0(r\xi)\,\mathcal{E}_1 + (\sqrt{\xi}Y_0(r\xi))'\,\mathcal{E}_2\big\}, \tag{6.91}$$

where $\mathcal{E}_j$ are the polynomials in r^{-2} of the degree $\leqslant 2$ (all other terms give $o(1)$ in the final result).

Firstly we consider the integral over r in the finite interval (T, P).

We write

$$rY_0(r\xi) = \frac{1}{\xi}\frac{\partial}{\partial r}rY_1(r\xi), \; \frac{\partial}{\partial \xi}\sqrt{\xi}Y_0(r\xi) = \sqrt{\frac{r}{\xi}}\frac{\partial}{\partial r}\sqrt{r}Y_0(r\xi) \tag{6.92}$$

and integrate by parts from T up to P.

The integrated terms at $r = T$ are exponentially small and at $r = P$ we have the main term

$$(c_1(N)+\frac{\Gamma'}{\Gamma}(1/2+iP))(2\xi^{-1}\tanh\xi/2)^{1/2}PY_1(P\xi)\mathcal{E}_1(P)q(P)\Omega(P)\tanh\pi P, \quad (6.93)$$

where $c_1(N)$ is the linear function in $\log N$.

Here

$$PY_1(P\xi) = -\frac{\partial}{\partial\xi}Y_0(P\xi) \quad (6.94)$$

and we can integrate by parts over ξ. The integrated terms at $\xi = 0$ and at $\xi = \infty$ are zeros (due to $\psi(Nl^2\sinh^2\xi/2)$ in the integrand) and the new integral will contain $Y_0(P\xi) = O(\frac{1}{\sqrt{P\xi}})$. So this term give zero at the limit $P \to +\infty$ and it rests estimate the convergent integral

$$\int_T^\infty\int_0^\infty rY_1(r\xi)\frac{\partial}{\partial r}\{(c_1(N)+\frac{\Gamma'}{\Gamma}(1/2+ir)+\frac{\Gamma'}{\Gamma}(1/2-ir))\mathcal{E}_1q\Omega\tanh\pi r\}\times$$
$$A_{01}(u.\frac{1}{\sinh\xi/2};1/2,\nu)\psi(Nl^2\sinh^2\xi/2)(\sinh\xi/2)^{-2iT_0}(\frac{\sinh\xi}{2\xi})^{1/2}\,d\xi\,dr \quad (6.95)$$

(we change the variable x in the definition (6.54) by $\sinh\xi/2$).

Here the derivative in respect to r equals $\frac{2}{r}-\frac{a_1}{r^3}+...$, so after one additional integration by parts we come to the expression

$$\xi^{-1}Y_0(r\xi)\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\{(c_1(N)+\frac{\Gamma'}{\Gamma}(1/2+ir)+\frac{\Gamma'}{\Gamma}(1/2-ir))\mathcal{E}_1q\Omega\tanh\pi r\} =$$
$$= \xi^{-1}Y_0(r\xi)(\frac{4a_1}{r^3}+...). \quad (6.96)$$

Since the kernel A_{01} is bounded here (for the real u and $\nu = 1/2+it, t \gg 1$), the result integration over ξ is not larger than

$$\frac{\log T}{T^2}\int_0^\infty\psi(Nl^2\sinh^2\eta/2)\frac{d\xi}{\xi} \ll \frac{\log T}{T^3}. \quad (6.97)$$

Exactly by the same way we estimate the integral with $(\sqrt{\xi}Y_0(r\xi))'$; using the second equality from (6.92) and integrating by parts we come to the integral with integrand

$$\sqrt{\frac{r}{\xi}}Y_0(r\xi)\frac{\partial}{\partial r}\{\sqrt{r}(c_1(N)+\frac{\Gamma'}{\Gamma}(1/2+ir)+\frac{\Gamma'}{\Gamma}(1/2-ir))\mathcal{E}_2q\Omega\tanh\pi r\}. \quad (6.98)$$

Here $\xi^{-1}\mathcal{E}_2 = \frac{b_1(\xi)}{\xi r^2} + \frac{b_2(\xi)}{\xi r^4} + ...$,so we have the considered integral with the additional multiplier r^2 in the denominator.

It is obviously that the same estimate is valid for the second integral in (6.66) for the case $f = F_{n,l}$.

6.6. The union of estimates.

Lemma 6.6. *Let ρ, ν and the initial function h are taken with conditions of subsection 3.1; then we have*

$$\int\limits_{(1/2)} \big(Z^{(d)}(\rho,\nu;1/2,1/2)|h_0,h_1) + Z^{(c)}(\rho,\nu;1/2,1/2|h_0+h_1\big)\omega_T(\rho)\,d\rho =$$

$$O(T^{1+\varepsilon}) + \{\textit{contribution of poles } \mathcal{Z}(\rho;\nu,1/2+ir)\} \quad (6.99)$$

For our specialization the coefficient h_1 is exponentially small for $\varkappa_j \leqslant T_0 - 2T^{1+\varepsilon}$ (lemmas 6.5 and 6.4 together with representation (4.17)).

The explicit averaging and the replacement of the inner sum over large $\varkappa_j$ by the convolution gives for this sum the estimate (the main term of this convolution is (6.80))

$$T\ \ O\Big(\sum_{N\leqslant N_0}\frac{d^2(N)}{N}\Big|\int A_{01}(u,\sqrt{N};1/2,\nu)h(u)\,d\chi(r)\Big|\Big) \ll T\log^4 T. \qquad (6.100)$$

All other terms from the convolution give $O(\frac{\log^2 T}{T^3}\sum_{N\leqslant N_0}\frac{d^2(N)}{N}) \ll T^{-3+\varepsilon}$ (the union of (6.76),(6.79),(6.86),(6.88) and (6.90)). So the main contribution gives the sum with the coefficient h_0.

For this sum we have (lemmas 6.2, 6.2) the estimate

$$\Big|\int\limits_{(1/2)} Z^{(d)}(\rho,\nu;1/2,1/2|h_0,0)\omega_T(\rho)\,d\rho\Big| \ll$$

$$\ll \log^2 T\sum_{\varkappa_j\leqslant R}\alpha_j\mathcal{H}_j^2(1/2) + T\sum_{R\leqslant\varkappa_j\leqslant 2T_0}\alpha_j\frac{\log^2\varkappa_j}{\varkappa_j^2}\mathcal{H}_j^2(1/2) \quad (6.101)$$

with $R = T_0^{1/2+\varepsilon}$. Using the asymptotic formula (58) from [8] we see that the right side is $O(T^{1+\varepsilon})$ for any fixed $\varepsilon > 0$.

6.7. The contribution of the poles $\mathcal{Z}(\rho;\nu,1/2)$.

Estimating the integrals

$$\int\limits_{(1/2)} Z^{(c)}(\rho,\nu;1/2,\,1/2|h^j)\omega_T(\rho)\,d\rho =$$

$$= \frac{1}{\pi}\int\int \mathcal{Z}(\rho;\nu,1/2+ir)\mathcal{Z}(1/2;1/2,1/2+ir)\frac{h_j(r)}{|\zeta(1+2ir)|^2}\omega_T(\rho)\,dr\,d\rho, \quad (6.102)$$

we can assume that for the case $j=0$ the integration over r have been done on the segment $|r|\leqslant 2T_0$ and for the case $j=1$ – on the line segment $r\geqslant T_0-2T^{1+\varepsilon}$, since h_0 is exponentially small for $r\geqslant 2T_0$ and h_1 is small for $r\leqslant T_0-2T^{1+\varepsilon}$.

For the similar reason we can assume that the integration over ρ have been done on the segment $(1/2+iT_0-iT_1,\,1/2+iT_0+iT_1)$ with $T_1=T^{1+\varepsilon}$.

Integrating over ρ in the first line we change this segment by the new path $\mathcal{C}$ which made up of
$\mathcal{C}_1$: segment $(1/2+iT_0-iT_1,\,2+iT_0-iT_1), T_1=T^{1+\varepsilon}$;
$\mathcal{C}_2$: segment $(2+iT_0-iT_1,\,2+iT_0+iT_1)$;
$\mathcal{C}_3$: segment $(2+iT_0+iT_1,\,1/2+iT_0+iT_1)$.

Two integrals over $\mathcal{C}_j$ with $j=1$ and $j=3$ are exponentially small; the integral over $\mathcal{C}_2$ have been considered early.

Now we consider the contribution of four residues of the integrand at the points $\rho=\nu+1/2\pm ir$ and $\rho=3/2-\nu\pm ir$ where $\mathcal{Z}(\rho;\nu,1/2+ir)$ has the simple pole.

These residues are the following integrals:

$$\zeta(2\nu)\int\limits_{|t\pm r-T_0|\leqslant T_1}\zeta(2\nu\pm 2ir)\zeta(1\pm 2ir)\frac{|\zeta(1/2+ir)|^4}{|\zeta(1+2ir)|^2}\times$$
$$\times\, h_j(r;1/2,\nu;\nu+1/2\pm ir,1/2)\omega_T(\nu+1/2\pm ir)\,dr \quad (6.103)$$

and the same integrals with ν replaced by $1-\nu$.

Using the known estimates of zeta-function on the unit line we see that these integrals are not larger than

$$o((\log T)^{10})\int_{T_0-T_1}^{T_0+T_1}|\zeta^4(1/2+ir)h_j(r;1/2,\nu;\nu+1/2\pm ir,1/2)\omega_T(\nu+1/2\pm ir)|\,dr. \quad (6.104)$$

It follows from the initial representations (2.19) and (2.20) that both functions $h_j(r;1/2,\nu;\nu+1/2\pm ir,1/2)$ are bounded for $T_0-T_1\leqslant r\leqslant T_0+T_1$. So we have the fourth moment of zeta-function and all these integrals are $O(T^{1+\varepsilon})$.

§7. SUM OVER REGULAR CUSP FORMS

7.1.The integral representations for the coefficients.

Lemma 7.1. *Let the coefficient $g(k)$ be defined by the equality (2.21) with the choice (3.1) for the parameters s, ν, ρ, μ and with Φ from (3.3) instead of Φ_N in (2.21);then we have*

$$g(k) \equiv g(k;\nu,\rho) = (-1)^{k-1} g_0(k;\nu,\rho) + g_1(k;\nu,\rho), \tag{7.1}$$

where

$$\begin{aligned} g_0(k;\nu,\rho) = \frac{2k-1}{\pi} \int_0^\infty & A_{00}(i(k-1/2), \sqrt{x}; 1/2, 1/2) \times \\ & \times \Big(\int_{-\infty}^{\infty} A_{00}(u, \frac{1}{\sqrt{x}; 1/2, \nu}) h(u)\, d\chi(u) - \\ & - \sum_{l \in L} (-1)^l c(l) A_{00}(i(l-1/2), \frac{1}{\sqrt{x}}; 1/2, \nu) \Big)\, x^{-\rho-1/2}\, dx \end{aligned} \tag{7.2}$$

and

$$\begin{aligned} g_1(k;\nu,\rho) = & \frac{2k-1}{i\pi} \cosh \pi t \times \\ \times \int_{(1/2)} & 2^{2w-1} \frac{\Gamma(k-1/2+w)}{\Gamma(k+1/2-w)} \gamma(\rho-w,\nu)\gamma(1/2-w,1/2) \hat{\Phi}(2w-2\rho+1)\, dw \end{aligned} \tag{7.3}$$

The first representation follows from the equalities

$$\begin{aligned} \frac{1}{2\pi} \int_0^\infty & A_{00}(-i(k-1/2), \sqrt{x}; 1/2, 1/2) x^{-w-1}\, dx = \\ & = \gamma(w,k)\gamma(1/2-w,1/2) \cos \pi w \sin \pi w, \end{aligned} \tag{7.4}$$

$$\begin{aligned} \frac{1}{2\pi} \int_0^\infty & A_{00}(u, \frac{1}{\sqrt{x}}; 1/2, \nu) x^{-\rho+w-1/2}\, dx = \\ & = \gamma(\rho-w,\nu)\gamma(w-\rho+1/2, 1/2+iu) \cos \pi(\rho-w) \cos \pi(w-\rho+1/2); \end{aligned} \tag{7.5}$$

here $k \geqslant 2$ is an integer and the last equality holds for $u = -i(l-1/2)$ if $l \geqslant 1$ be an integer.

The function g_1 is the part of (2.21) with $s = \mu = 1/2$.

7.2. The coefficient g_1.

Proposition 7.1.

$$|g_1(k;\nu,\rho)| \ll k^{-3}e^{-\tau}. \tag{7.6}$$

Really, the integrand in (7.5) on the line $\mathrm{Re}\, w = -3/2$ is $O(k^{-4}\exp(-\pi(\tau + 2|w|))\,|\rho - w|^5)$ for $\mathrm{Im}\, w < 0$, $O(k^{-4}e^{-\pi\tau}\tau^5)$ for $0 \leqslant \mathrm{Im}\, w \leqslant \tau - t$ and $O(k^{-4}e^{-\pi(|w|+t)}|w|^5)$ for $\mathrm{Im}\, w > \tau - t$.

The sum over regular cusp forms with this coefficient g_1 gives $o(1)$ into the final result.

7.3. The kernel $A_{00}(i(k-1/2), x; 1/2, 1/2)$.

We substitute in (4.8) $r = i(k-1/2)$ with an integer k,$\nu = 1/2$ and $s = 1/2+\varepsilon$; taking the limit $\varepsilon \to 0$ we come to the definition

$$A_{00}(i(k-1/2), x; 1/2, 1/2) = 2x(\frac{\Gamma'}{\Gamma}(1) - \frac{\Gamma'}{\Gamma}(k))F(k, 1-k; 1; 1-x^2) -$$
$$- x\frac{\partial}{\partial\varepsilon}(|1-x^2|^{\varepsilon}F(k+\varepsilon, 1-k+\varepsilon; 1+2\varepsilon; 1-x^2))\Big|_{\varepsilon=0} \tag{7.7}$$

It follows from this equality that two functions

$$v = \sqrt{2\tan\xi/2}A_{00}(i(k-1/2), \cos\xi/2; 1/2, 1/2),$$
$$\tilde{v} = \sqrt{2\tanh\xi/2}A_{00}(i(k-1/2), \cosh\xi/2; 1/2, 1/2) \tag{7.8}$$

are the solutions of the differential equations

$$v'' + ((k-1/2)^2 + \frac{1}{4\sin^2\xi})v = 0, \tilde{v}'' + (-(k-1/2)^2 + \frac{1}{4\sinh^2\xi})\tilde{v} = 0. \tag{7.9}$$

Lemma 7.2. *Let k be sufficiently large; then for $0 \leqslant \xi \leqslant 3\pi/4$ we have the asymptotic series*

$$- \pi\sqrt{2\tan\xi/2}A_{00}(i(k-1/2), \cos\xi/2; 1/2, 1/2) =$$
$$= \sqrt{\xi}\, Y_0((k-1/2)\xi)\sum_{n\geqslant 0}\frac{a_n(\xi)}{(k-1/2)^{2n}} +$$
$$(\sqrt{\xi}\, Y_0((k-1/2)\xi))'\sum_{n\geqslant 1}\frac{b_n(\xi)}{(k-1/2)^{2n}}, \tag{7.10}$$

where $a_0 \equiv 1$ and all a_n, $\xi^{-1}b_n$ with $1 \leqslant n \leqslant M$ are bounded smooth functions for any fixed M;furthermore, we have the expansion of the same form for $(2\tan\xi/2)^{-1/2}A_{00}(i(k-1/2), \sin\xi/2; 1/2, 1/2), 0 \leqslant \xi \leqslant 3\pi/4$.

Lemma 7.3. *Let k be sufficiently large;then for $\xi \geqslant 0$ we have the uniform asymptotic formula*

$$\sqrt{2\tan\xi/2}A_{00}(i(k-1/2),\cosh\xi/2;1/2,1/2)=2\sqrt{\xi}K_0((r-1/2)\xi)(1+O(\frac{1}{k})). \tag{7.11}$$

The proofs may be omitted for both lemmas since these ones are exactly the same what we had in §5.

7.4.The coefficients $g_0(k)$ with large k's.

Lemma 7.4. *Let $k \geqslant k_0(T) = T^4$; then we have*

$$\int\limits_{(1/2)} \omega_T(\rho)\sum_{k\geqslant k_0} g_0(k)\sum_j \alpha_{j,2k}\mathcal{H}_{j,2k}(\rho+it)\mathcal{H}_{j,2k}(\rho-it)\mathcal{H}^2_{j,2k}(1/2)\,d\rho \ll T_0 \tag{7.12}$$

To get this estimate we use the initial definition (2.21); we integrate on the line $\Delta = \operatorname{Re} w = -2$. The integrand on this line is not larger than

$$|k+w|^{-5}\left(|w-\rho|+1\right)^{-6}\left(|w|+1\right)^4$$

(we use the Stirling expansion and the estimate $|\hat{\Phi}(2w)| \ll |w|^{2\operatorname{Re} w-6}$).

Consequently, we have

$$|g_0(k)| \ll \frac{{T_0}^4}{k^4} \tag{7.13}$$

It follows from the functional equation that for $|\rho| = o(k)$

$$|\mathcal{H}_{j,2k}(\rho+it)\mathcal{H}_{j,2k}(\rho-it)| \ll k^2. \tag{7.14}$$

It gives us the bound

$$\ll \sum_{k\geqslant k_0}\frac{T_0^5}{k^2}\sum_j \alpha_{j,2k}\mathcal{H}^2_{j,2k} \tag{7.14}$$

for the sum (7.12). It is known [10] that

$$\sum_{k\leqslant L}\sum_j \alpha_{j,2k}\mathcal{H}^2_{j,2k}(1/2) \ll L\,\log L \tag{7.16}$$

and we come to (7.12).

7.5. Sum with $k \leqslant {T_0}^4$.

Proposition 7.2.

$$\frac{1}{4i}\int\limits_{(1/2)} \omega_T(\rho) Z^{(r)}(\rho,\nu;1/2,1/2|g_0)\,d\rho =$$

$$= \sum_{N,l\geqslant 1}\sum_{k\leqslant T_0^4} G(k,N,l)\sum_j \alpha_{j,2k}\frac{t_{j.2k}(N)\tau_\nu(N)}{(Nl^2)^{1/2+iT_0}}\mathcal{H}^2_{j,2k}(1/2)+O(T_0) \quad (7.17)$$

where ψ is defined by (6.14) and

$$G(k,N,l)=\frac{(2k-1)T}{\pi}\int_0^\infty A_{00}(i(k-1/2),\sqrt{x};1/2,1/2)\times$$
$$\times\left(\int_{-\infty}^\infty A_{00}(u,\frac{1}{\sqrt{x}};1/2,\nu)h(u)\,d\chi(u)-\right.$$
$$\left.-\sum_{l\in L}(-1)^l c(l)A_{00}(-i(l-1/2),\frac{1}{\sqrt{x}};1/2,\nu)\right)\psi(Nl^2x)x^{-1/2-iT_0}\,dx \quad (7.18)$$

This representation follows from (7.2) by the same way as in Proposition 6.2; the part of $Z^{(r)}$ with $k \geqslant T_0^4$ gives $O(T_0)$.

7.5.1. Case $Nl^2 \geqslant 2$.

Lemma 7.5. *For any $\varepsilon>0$ the part of the sum in (7.17) with $Nl^2\geqslant 2$ is not larger than $O(\exp(-t)+\exp(-T^\varepsilon))$.*

Really, the function $\psi(x)$ is $O(\exp(-(NT)^\varepsilon))$ if $|x-1|\geqslant (NT)^\varepsilon T^{-1}$. So the part of our integral with $|Nl^2x-1|\geqslant \frac{(NT)^\varepsilon}{T}$ gives the contribution $O(\exp(-T^\varepsilon))$ into (7.17).

Under the condition $Nl^2\geqslant 2$ we have in the remaining part $x\leqslant 1/2$; but for these values of x we have (see (5.62))

$$|A_{00}(u,\frac{1}{\sqrt{x}};1/2,\nu)| \ll \exp(-\pi t/2)$$

and the same estimate is valid if $u=-i(l-1/2)$ with the fixed integer l.

So for $N \ll T^B$ for some fixed B we have for our sum the estimate $O(e^{-t})$. But for $N \gg T^B$ we can integrate over the variable u on the line $\operatorname{Im} u = -3/2$ when the integrand contains $I_{2iu}(t\xi)$(here $\sin \xi/2 = \sqrt{x}$; in terms with I_{-2iu} we change u by $-u$). For $\xi \ll (Nl^2)^{-1/2}$ it gives the inequality

$$|\int_{-\infty}^{\infty} A_{00}(u, \frac{1}{\sin \xi/2}; 1/2, \nu)\, h(u)\, d\chi(u)| \ll \frac{t^3}{(Nl^2)^{3/2}}\, e^{-\pi t}. \tag{7.20}$$

It means the series in N, l is convergent and we get $O(e^{-t})$ again.

7.5.2. Case $N = l = 1$.

Lemma 7.6. *For any fixed $\varepsilon > 0$ we have*

$$|G(k, 1, 1)| \ll \begin{cases} T\, k^{-1} \log^2 k, \ k \geqslant T^{1/2+\varepsilon}, \\ k\, \log^2 T, \ k \leqslant T^{1/2+\varepsilon}. \end{cases} \tag{7.21}$$

It is sufficient estimate that part of integral where $|x - 1| \ll \frac{\log T}{T}$.

Doing the change of the variable $x \mapsto \cos^2 \xi/2$ for $x \leqslant 1$ and $x \mapsto \cosh^2 \xi/2$ for $x \geqslant 1$ and using the asymptotic formulas (5.67),(5.59),(7.11) and (7.10) we come to the integral with the main term

$$\int_0^\delta Y_0((k - 1/2)\xi) K_0(t\xi) \psi(\cos^2 \xi/2)(\cos \xi/2)^{-2iT_0} \xi\, d\xi +$$
$$+ \frac{1}{2} \int_0^\delta K_0((k - 1/2)\xi) Y_0(t\xi) \psi(\cosh^2 \xi/2)(\cosh^2 \xi/2)^{-2iT_0} \xi\, d\xi \tag{7.22}$$

where $\delta = \delta(T) = \frac{\log T}{\sqrt{T}}$.

In this integral $K_0(t\xi)$ and $Y_0(t\xi)$ may be replaced by the corresponding power series since $t\xi \ll T^{-1/4}$.

If $k\delta(T) \ll T^\varepsilon$ both integrals here are $O(\frac{\log^2 T}{T})$; it gives the second inequality in (7.21).

Let now $k\delta(T) \gg T^\varepsilon$. The second integral in (7.22) is

$$O(\int_0^\infty K_0((k - 1/2)\xi) \xi (|\log \xi| + 1)\, d\xi) = O(\frac{\log k}{k^2}).$$

In the first integral the small interval $\xi \leqslant k^{\varepsilon - 1}$ contributes $O(\frac{\log^2 k}{k^2})$.

If $k^{\varepsilon-1} \leqslant \xi \leqslant \delta(T)$ we use the asymptotic expansion for $Y_0((k-1/2)\xi)$. After that we have the integrals of the form

$$\frac{1}{\sqrt{k}} \int \exp(\pm i(k-1/2)\xi - 2iT_0 \log\cos\xi/2 + \log\psi(\cos^2\xi/2))\sqrt{\xi}\, d\xi. \tag{7.23}$$

The point of the stationary phase is near to $(2k-1)T_0^{-1}$; for $k \geqslant T^{1/2+\varepsilon}$ this point lies outside of the interval $(0, \delta(T))$. So we can integrate by parts any times and this part of our integral is $O(k^{-2}\log^k)$ also.

7.5.3.Sum over regular cusps.

Lemma 7.7. *For any fixed $\varepsilon > 0$ we have for our specialization ((3.1), (3.2) and the conditions for h in 3.1)*

$$\int\limits_{(1/2)} \omega_T(\rho) Z^{(r)}(\rho,\nu; 1/2, 1/2|g_0)\, d\rho \ll T^{1+\varepsilon}. \tag{7.24}$$

This estimate is the direct consequence of (7.17), (7.21), Lemma 7.5 and (7.16). The integral on the left side (7.24) is not larger than

$$\sum_{k\leqslant k_0} k \log^2 T \sum_j \alpha_{j,2k}\mathcal{H}^2_{j,2k}(1/2) + \sum_{k_0\leqslant k\leqslant T^4} \frac{T}{k} \sum_j \alpha_{j,2k}\mathcal{H}^2_{j,2k}(1/2) \ll T^{1+2\varepsilon}\log^3 T \tag{7.25}$$

(it follows from (7.16)); this inequality concludes the proof.

§8. $\mathcal{R}$ AND $\mathcal{R}_h$ – FUNCTIONS

To finish the estimation it rests consider the average of 12 terms with $\mathcal{R}_h$ and $\mathcal{R}$ on the right side (2.57) for the case of our specialization.

Here some terms have the poles at $\mu = 1/2$ and $s = 1/2$, but the full sum must be regular since all other sums and integrals in this identity are regular at this point.

8.1. Terms without pole at $s = 1/2$.

Firstly we consider the simplest case when there are no poles at $s = 1/2$.

Proposition 8.1. *For $\nu = 1/2 + it, \rho = 1/2 + i\tau,\ t, \tau \to \infty, t = o(\tau^{1/4})$ we have*

$$\mathcal{R}_h(1/2, \nu; \rho, 1/2) \equiv 2(4\pi)^{-2\nu}\hat{\Psi}(2\nu)\frac{\zeta(2\rho)\zeta(2\nu)}{\zeta(2\rho+2\nu)}\zeta^4(\rho+\nu) = O(t^{-6}\log^6\tau) \quad (8.1)$$

This equality is the result of the substitution $s = \mu = 1/2$ in the definition (2.18); after that we use the trivial estimate $\zeta(1+it) = O(\log t)$.

Proposition 8.2. *For any fixed $M \geqslant 2$ we have*

$$\mathcal{R}(1/2, \nu; \rho, 1/2|h) = -\frac{\zeta(2\nu)\zeta^2(\rho+\nu)\zeta^2(\rho-\nu)}{\zeta(1+2\nu)}\,h(-i\nu) = O(\frac{1}{t^M}). \quad (8.2)$$

It follows from the definition and our assumptions about h.

Proposition 8.3. *For any fixed $M \geqslant 2$ we have*

$$\lim_{\mu\to 1/2}\Bigg(\mathcal{R}(\rho, \mu; 1/2, \nu|h) + \mathcal{R}(\rho, 1-\mu; 1/2, \nu|h)\Bigg) = O(\tau^{-M}) \quad (8.3)$$

Writing $\zeta(2\mu) = \frac{1}{2\mu-1} + \gamma + ...$ (here γ is the Euler constant) we see that the left side on (8.3) equals to

$$2\gamma\, h(i(\rho-1))\frac{\zeta(2\rho-1)}{\zeta(3-2\rho)}\zeta(\rho-\nu)\zeta(1-\rho+\nu)\zeta(\rho+\nu-1)\zeta(2-\rho-\nu)+$$
$$+ 2\zeta(2\rho-1)\frac{\partial}{\partial\mu}\frac{h(i(\rho-\mu-1/2))}{\zeta(2-2\rho+2\mu)}\mathcal{Z}(1/2;\nu,\rho-\mu)|_{\mu=1/2}$$

Now the assertion follows since $h(r)$ decreases more rapidly than any fixed degree of r.

Proposition 8.4. *Under our assumptions for h we have for any fixed $M \geqslant 4$*

$$\mathcal{R}(1/2, \nu; \rho, 1/2) = -\frac{\zeta(2\nu)\zeta^2(\rho+\nu)\zeta^2(\rho-\nu)}{\zeta(2\nu+1)}\,h(-i\nu) = O(t^{-M}). \quad (8.4)$$

It is the consequence of the definition and the fast decreasing of h.

Proposition 8.5. *Let $\tilde{h}_0(r), \tilde{h}_1(r)$ are the values of two integrals (2.19) and (2.20) for the case* $\operatorname{Im} r < -\Delta$*; then for* $-1-\Delta < r < -\Delta$ *we have the analitical continuation*

$$h_0(r) = \tilde{h}_0(r) + 2^{2ir}\Gamma(2ir)\gamma(\rho - ir, \nu)\gamma(s - ir, \mu)\cosh \pi r \times$$
$$\times \left(\cos \pi(\rho - ir)\cos \pi(s - ir) + \sin \pi\nu \sin \pi\mu\right) \hat{\Psi}(2ir - 2s - 2\rho + 2), \quad (8.5)$$

$$h_1(r) = \tilde{h}_1(r) + 2^{2ir}\Gamma(2ir)\gamma(\rho - ir, \nu)\gamma(s - ir, \mu)\cosh \pi r \times$$
$$\times \left(\cos \pi(s - ir)\sin \pi\nu + \cos \pi(\rho - ir)\sin \pi\mu\right) \hat{\Psi}(2ir - 2s - 2\rho + 2). \quad (8.6)$$

Really, these integrals (for $0 < \Delta < 1/2$) define the even regular functions in the strip $\operatorname{Im} r < \Delta$. Writing

$$\gamma(w, 1/2 + ir) = \frac{2^{2w-1}}{\pi} \frac{\Gamma(w + ir + 1)\Gamma(w - ir + 1)}{(w + ir)(w - ir)} \quad (8.7)$$

we come to the Cauchy type integral. We have two simple poles at $w = \pm ir$. When $\operatorname{Im} r$ is near to $-\Delta$ we deform the path of the integration so that the point ir lies to right on the path. When $\operatorname{Im} r < -\Delta$ we can take the initial line as the path of the integration; it gives us the relation

$$h_0(r) = \tilde{h}_0(r) + 2\pi i \operatorname{Res}_{w=ir}\left(-i(\text{the integrand})\right) \quad (8.8)$$

and we get (8.5) and (8.6).

Proposition 8.6. *Under assumptions of subsection 3.1 we have*

$$\mathcal{R}(\rho, \nu; 1/2, 1/2|h_0 + h_1) = O(t^{-5}\log^2 \tau), \quad (8.9)$$

$$\mathcal{R}(\rho, 1-\nu; 1/2, 1/2|h_0 + h_1) = O(t^{-5}\log^2 \tau). \quad (8.10)$$

First of all (we use (2.49) with $s = \mu = 1/2$)

$$\mathcal{R}(\rho, \nu; 1/2, 1/2|h_j) = 2\frac{\zeta(2\rho-1)\zeta(2\nu)}{\zeta(2-2\rho+2\nu)}\zeta^2(1-\rho+\nu)\zeta^2(\rho-\nu)\, h_j(i(\rho-\nu-1/2))$$
$$= O(\tau^{3/2}\log^2\tau)\,|h_j(i(\rho-\nu-1/2))|. \quad (8.11)$$

Now for h_j, $j = 0$ or $j = 1$, we use (8.5)–(8.6). In these equalities we have for difference $h_j - \tilde{h}_j$ the bound $O(\tau^{-3/2})|\hat{\Psi}(2\nu - 4\rho + 2)| = O(\tau^{-13/2})$ since $|\hat{\Psi}(2w)| \ll |w|^{2\mathrm{Re}w-6}$.

So it rests estimate the integrals $\tilde{h}_j$ at the point $r = i(\rho - \nu - 1/2)$.

We have

$$\tilde{h}_j(i(\rho-\nu-1/2)) = \frac{i}{\pi^2}\int_{(\Delta)} \frac{\Gamma(w+\rho-\nu-1/2)\Gamma(\rho-w+\nu-1/2)}{\cos\pi(\rho-w-\nu)}\Gamma^2(1/2-w)\times$$
$$\times S_j(w,\rho,\nu)\hat{\Phi}(2w-2\rho+1)\,dw, \quad (8.12)$$

where it is assumed $0 < \Delta < 1/2$ and

$$S_0 = -2^{2\rho-2w-4}\Bigg(\sin\pi(\rho+w) + \sin\pi(3w-\rho) + 2\sin\pi(\nu+w) + 2\sin\pi(\nu-w)\Bigg), \quad (8.13)$$

$$S_1 = 2^{2\rho-2w-4}\Bigg(\sin\pi(w+\rho-2\nu) + \sin\pi(w+2\nu-\rho) + \sin\pi(2\rho-\nu-w) -$$
$$- \sin\pi(w+\rho) - \sin\pi(w-\rho) + \sin\pi(w-\nu)\Bigg). \quad (8.14)$$

Firstly we consider the integral h_0.

Let us write $w = \Delta + i\eta$; then for $\rho = 1/2 + i\tau$, $\nu = 1/2 + it$ we have the following estimate for the integrand:

$$\ll \exp(-\frac{\pi}{2}\mathcal{M})\times$$
$$\times(|\eta+\tau-t|+1)^{\Delta-1}(|\tau-\eta+t|+1)^{-\Delta}(|\eta|+1)^{-2\Delta}(|\eta-\tau|+1)^{2\Delta-6} \quad (8.15)$$

where

$$\mathcal{M} = |\eta + \tau - t| + |\tau - \eta + t| + 2|\eta| - 2|\tau - \eta - t| - 2\max(|\tau - 3\eta|, |\tau + \eta|))$$

(it follows from the Stirling expansion and from our construction of Φ).

If $\eta \leqslant -2\tau$ or $\eta \geqslant 2\tau$ this estimate gives not larger than $O(|\eta|^{-7})$ and this part of our integral is $O(\tau^{-6})$.

Now for $-\tau + t \leqslant \eta \leqslant 0$ we have the exponential multiplier $\exp(-\pi(\tau - t + \eta))$, for $0 \leqslant \eta \leqslant \tau - t$ there is $\exp(-\pi(\tau - t - \eta))$ in the integrand, for $\tau - t \leqslant \eta \leqslant \tau$ this multiplier is $O(\exp(-\pi(\eta - \tau + t)))$ and for $\tau \leqslant \eta \leqslant \tau + t$ we have the multiplier $\exp(-\pi(\tau + t - \eta))$. This exponential multiplier equals to 1 for $\eta \leqslant -\tau + t$ and for $\eta \geqslant \tau + t$; it is exponentially small for $\eta \in (-\tau + t, \tau + t)$. So the parts of our integral with $-\tau + t + 2\log\tau \leqslant \eta \leqslant \tau - t - 2\log\tau$ or $\tau - t + 2\log\tau \leqslant \eta \leqslant \tau + t - 2\log\tau$ give the small contribution. The part of this integral with $\eta \approx -\tau$ is $O(\tau^{-5})$.

Finally, for $\eta \approx \tau$ we move the path of integration on the line $\operatorname{Re} w = 2$. The residues at $w = \rho \pm (\nu - 1/2)$ give $O(\tau^{-3/2}\, t^{-11/2})$ and the result of integration over the line $\operatorname{Re} w = 2$ is $O(\tau^{-3}\, t^{-2})$.

We have the similar situation for the case $j = 1$. The integrand contains the multiplier $\exp(-\pi|\eta|)$ for $\eta \leqslant \frac{1}{2}(\tau - t)$, the multiplier $\exp(-\pi(\tau - t - \eta))$ for $\frac{1}{2}(\tau - t) \leqslant \eta \leqslant \tau + t$ and $\exp(-2\pi(\eta - \tau))$ for $\eta \geqslant \tau + t$.

It means this integral is determined by two intervals: $|\eta| \leqslant 2\log\tau$ and $|\eta - \tau + t| \leqslant 2\log\tau$. In the case $w \approx \rho - \nu$ we integrate over the line $\operatorname{Re} w = 2$ and the pole at $w = \rho - \nu + 1/2$ contributes $O(\tau^{-3/2} t^{-11/2})$ again.

Together with (8.11) the estimates of $|\tilde{h}_j|$ give (8.9) and (8.10).

8.2. Terms with pole at $s = 1/2$.

Proposition 8.7. *Under our assumptions for h we have*

$$\lim_{s,\nu \to 1/2} \Bigg(\mathcal{R}_h(\rho, \mu; s, \nu)) + \mathcal{R}_h(\rho, 1 - \mu; s, \nu) + \\ + \mathcal{R}(s, \mu; \rho, \nu | h_0 + h_1) + \mathcal{R}(s, 1 - \mu; \rho, \nu | h_0 + h_1 \Bigg) = O(t^{-5} \log^2 \tau) \quad (8.16)$$

Here two first terms have the pole at $s = 1/2$, since

$$\mathcal{R}_h(\rho, \mu; s, \nu) = 2\zeta(2s)(4\pi)^{2\rho - 2\mu - 1} \hat{\Phi}(2\mu + 1 - 2\rho) \frac{\zeta(2\mu)}{\zeta(2s + 2\mu)} \mathcal{Z}(s + \mu; \rho, \nu) \quad (8.17)$$

(sum $\mathcal{R}_h(\rho,\mu;s,\nu)+\mathcal{R}_h(\rho,1-\mu;s,\nu)$ has no pole at $\mu=1/2$).

To see that sum in (8.16) has no pole at $s=1/2$ we transform the expression for $\mathcal{R}$. By the definition,

$$\mathcal{R}(s,\mu;\rho,\nu|h_0+h_1)=2\frac{\zeta(2s-1)\zeta(2\mu)}{\zeta(2-2s+2\mu)}\mathcal{Z}(\rho;\nu,s-\mu)(h_0+h_1)(i(s-\mu-1/2)). \quad (8.18)$$

First of all we have (we use (8.6),(8.7) and the definition (1.31))

$$\begin{aligned}(h_0+h_1)(i(s-\mu-1/2)) &= (\tilde{h}_0+\tilde{h}_1)(i(s-\mu-1/2))+\\ &+2^{2\rho+4s-2\mu-4}\frac{\Gamma(2s-1)}{\pi\cos(s-\mu)}\Gamma(\rho+\nu+s-1)\Gamma(\rho-\nu+s-\mu)\times\\ &\times\big(\sin\pi(2s-\mu)+\sin\pi\mu)(\sin\pi(\rho+s-\mu)+\sin\pi\nu\big)\hat{\Phi}(2\mu+3-4s-2\rho) \quad (8.19)\end{aligned}$$

(the functional equation for gamma-function have been used here).

The last term has the pole at $s=1/2$ also; it compensates the pole from $\mathcal{R}_h$.

The following identity follows from the Riemann functional equation and the doubling formula for the gamma-function.

Proposition 8.8.

$$\begin{aligned}(\sin\pi(\rho+s-\mu)+\sin\pi\nu)\Gamma(\rho+\nu+s-\mu-1)\Gamma(\rho-\nu+s-\mu)\mathcal{Z}(\rho;\nu,s-\mu)=\\ =\frac{1}{2}(2\pi)^{2\rho+2s-2\mu-1}\mathcal{Z}(1-s+\mu;\rho,\nu) \quad (8.20)\end{aligned}$$

Really, changing zeta-functions of arguments $\rho-\nu+s-\mu$ and $\rho+\nu+s-\mu-1$ by the functions at $1-\rho+\nu+\mu-s$ and $2-\rho-\nu+\mu-s$ correspondingly we get

$$\begin{aligned}\mathcal{Z}(\rho;\nu,s-\mu) &= \zeta(\rho+\nu-s+\mu)\zeta(\rho-\nu+s-\mu)\zeta(\rho+\nu+s-\mu-1)\zeta(\rho-\nu-s+\mu+1)\\ &= \mathcal{Z}(1-s+\mu;\rho,\nu)\times\\ &\times\pi^{2\rho+2s-2\mu-2}\frac{\Gamma((1-\rho+\nu+\mu-s)/2)\Gamma((2-\rho-\nu+\mu-s)/2)}{\Gamma((\rho-\nu+s-\mu)/2)\Gamma((\rho+\nu+s-\mu-1)/2)} \quad (8.21)\end{aligned}$$

Now the doubling formula gives

$$\begin{aligned}\Gamma(\rho+\nu+s-\mu-1)\Gamma(\rho-\nu+s-\mu) &= \pi^{-1}2^{2\rho+2s-2\mu-3}\Gamma((\rho+\nu+s-\mu-1)/2)\\ &\times\Gamma((\rho+\nu+s-\mu)/2)\Gamma((\rho-\nu+s-\mu)/2)\Gamma((\rho-\nu+s-\mu+1)/2) \quad (8.22)\end{aligned}$$

and we have (using the functional equation for gamma-function again)

$$\Gamma(\rho+\nu+s-\mu-1)\Gamma(\rho-\nu+s-\mu)\mathcal{Z}(\rho;\nu,s-\mu)=$$
$$=\frac{(2\pi)^{2\rho+2s-2\mu-1}}{\sin\pi(\rho+\nu+s-\mu)/2\ \sin\pi(\rho-\nu+s-\mu+1)/2}\mathcal{Z}(1-s+\mu;\rho,\nu); \quad (8.23)$$

this equality coincides with (8.20).

We see now that the coefficient in front of $\zeta(2s)$ equals to

$$2\frac{\zeta(2\mu)}{\zeta(1+2\mu)}\left((4\pi)^{2\rho-2\mu-1}\hat{\Phi}(2\mu+1-2\rho)\mathcal{Z}(\mu+1/2;\rho,\nu)+(s-1/2)r_1(\mu,\nu)...\right) \quad (8.24)$$

and the coefficient in front of $\Gamma(2s-1)$ equals to (note that $\zeta(0)=-1/2$)

$$2\frac{\zeta(2\mu)}{\zeta(1+2\mu)}\Bigg(-(4\pi)^{2\rho-2\mu-1}\hat{\Phi}(2\mu+1-2\rho)\mathcal{Z}(\mu+1/2;\rho,\nu)\sin\pi\mu+$$
$$+(s-1/2)r_2(\mu,\nu)...\Bigg), \quad (8.25)$$

where $r_1(\mu,\nu)$, $r_2(\mu,\nu)$ are regular at $\mu=1/2$.

It means there is the limit $s\to 1/2$ (8.16) and it has the form

$$\zeta(2\mu)\mathcal{F}(\mu,\rho)+\zeta(2-2\mu)\mathcal{F}(1-\mu,\rho);$$

here $\mathcal{F}(\mu,\rho)$ is regular at $\mu=1/2$ and contains the product of four zeta-functions on the unit line (and the derivative of this product).

Now the existence of the limit $\mu\to 1/2$ is obvious; there is no need to calculate the explicit form of this limit. One can see that the result contains the derivatives of zeta-functions of the order $\geqslant 2$ and the derivatives of $\hat{\Phi}$ at the point $2-2\rho$ of the order 0,1,2. Since these last derivatives have the same order as $\hat{\Phi}$ (each differentiation gives the additional multiplier $O(\log\tau)$ only) we get the estimate $O(\tau^{-6}\log^6\tau)$ for this limit.

It rests estimate two terms with $(\tilde{h}_0+\tilde{h}_1)(i/2)$. We have $\tilde{h}_1(i/2)=0$ and (for $s=\mu=1/2$)

$$\tilde{h}_0(i/2)=\frac{2^{2\rho-2w-1}}{i\pi}\int\limits_{(\Delta)}\frac{\cos\pi(\rho-w)\sin\pi w+\sin\pi\nu}{(2w-1)\cos\pi w}\times$$
$$\times\Gamma(\rho-w+\nu-1/2)\Gamma(\rho-w-\nu+1/2)\hat{\Phi}(2w-2\rho+1)\,dw. \quad (8.26)$$

The part of this integral with $|w| \leqslant (1-\delta)\,\tau$ with some fixed (small) $\delta > 0$ is $O(\tau^{-6}\log\tau)$. For $\eta \geqslant (1-\delta)\,\tau$ (here $\eta = \operatorname{Re} w$) we have

$$
\begin{aligned}
|\tau-\eta+t| + |\tau-\eta-t| - 2|\tau-\eta| &= 0, \ (1-\delta)\tau \leqslant \eta \leqslant \tau - t,\\
&= 2(\eta-(\tau-t)), \ \tau - t \leqslant \eta \leqslant \tau,\\
&= 2(\tau+t-\eta), \ \tau \leqslant \eta \leqslant \tau+t,\\
&= 0, \qquad \eta \geqslant \tau + t.
\end{aligned}
$$

It means thar our integral is determined by the interval $|\tau - t - \eta| \leqslant 2\log\tau$ ang by halfaxis $\eta \geqslant \tau + t - 2\log\tau$. For this η we can integrate over the line $\Delta = 0$; since for both cases $|w-\rho| \geqslant t$ we have

$$
|\tilde{h}_0(i/2)| \ll \tau^{-1}\, t^{-5}. \tag{8.27}
$$

Taking into account that

$$
|\mathcal{Z}(\rho;\nu,0)| \ll \tau\ \log^2\tau, \tag{8.28}
$$

we come to the final estimate (8.16).

§9. THE LINDELÖF CONJETCTURE

It rests collect together all previous estimates to finish the proof of (0.2) and (0.3).

As it had been shown in subsection 3.1 the result of the averaging of the left side of our main functional equation (2.57) for the special case $\rho = 1/2 + i\tau, \nu = 1/2 + it, s = \mu = 1/2$ for any positive (small) δ and for any fixed $M \geqslant 1$ is not smaller than

$$
\begin{aligned}
T\Biggl(\sum_{j\geqslant 1} \alpha_j |\mathcal{H}_j(\nu)|^2\, h(\varkappa_j) +\\
\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\zeta(\nu+ir)\zeta(\nu-ir)|^2}{|\zeta(1+2ir)|^2} h(r)\, dr\Biggr) + o(1) \gg\\
\gg T/4 \sum_{1\leqslant j\leqslant M} \alpha_j |\mathcal{H}(\nu)|^2 h(\varkappa_j) + \frac{T}{4\pi}\int_{-\delta}^{\delta} \frac{|\zeta(\nu+ir)\zeta(\nu-ir)|^2}{|\zeta(1+2ir)|^2}\, h(r)\, dr
\end{aligned} \tag{9.1}
$$

If $\delta \ll (\log t)^{-2}$ the last integral is not smaller than

$$(h(0)/3\pi)\, T\, \delta^3 |\zeta(1/2+it)|^4. \tag{9.2}$$

On the other side this average is not larger than

$$\log^2 T \sum_{\varkappa_j \leqslant \sqrt{T}_0} \alpha_j \mathcal{H}_j^2(1/2) + T \sum_{\sqrt{T}_0 \leqslant \varkappa_j \leqslant T_0^4} \alpha_j \frac{\log^2 \varkappa}{\varkappa^2} \mathcal{H}_j^2(1/2) + T^{1+\varepsilon} +$$

(Lemmas 6.1, 6.2, 6.6)

$$+\log^2 T \sum_{k \leqslant k_0} k \sum_j \alpha_{j,2k} \mathcal{H}_{j,2k}^2(1/2) + T \sum_{k_0 \leqslant k \leqslant k_1} |\frac{\log^2 k}{k} \sum_j \alpha_{j,2k} \mathcal{H}_{j,2k}^2(1/2) +$$

(Lemma 7.6; here $k_0 = T_0^{1+\varepsilon}, k_1 = T^4$)

$$+O(T\frac{\log^5}{t^5})+$$

(Propositions 8.6, 8.7)

+{terms of the smaller order}.

All listed sums are $O(T^{1+\varepsilon})$ for any fixed $\varepsilon > 0$. Consequently (we take $\delta = (\log t)^{-2}$ in (9.2)),

$$|\zeta^4(1/2+it)| \ll T^\varepsilon \tag{9.3}$$

and for every fixed $j \geqslant 1$

$$|\mathcal{H}_j(1/2+it)|^2 \ll T^\varepsilon. \tag{9.4}$$

The unique condition for t is $t = o(T^{1/4})$; so, supposing $t = T^{1/8}$ we come to (0.2) and (0.3).

§10. SOME CONCLUDING REMARKS

I had been writing this work many years. The plans of the proof had been changed many times. Some parts had retained, the others were written anew.

Now the potential reader has the definitive text. May be, it would be useful to indicate the crucial points of the proposed proof.

Of course, the base of the whole construction is the the fore-trace formulas (1.13) and (1.25) which are well known today.

The first crucial point is the regularization (2.11) where the coefficients are defined by (2.10). This simple trick took off many problems with the convergence and it is hardly understand why this method was not used early.

The idea of the special averaging of the main functional equation (2.61) (when the unknown quantity – namely, $|\zeta(1/2+it)|^4$ – is the coefficient in front of large parameter) is blowed by the proof of the Dirihlet formula for number of classes.

To realize this idea the new representations (4.17) are necessary (together with the initial expressions (2.19) and (2.20)).

The whole technical §5 must be outside of this work. This section may be considered as the supplement to the Erdélyi-Bateman handbook; one can compare our approach with text on the pages 602-643 in [14].

The real crucial step is the replacement of the summation over the discret spectrum (with very large $\varkappa_j$) by the integration over this variable. The replacement of the sum of the quatities (6.52) over the full spectrum by the sum (6.72) allows us use my convolution formulas and, as the result, to finish estimates.

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